MATH 241 Spring, 2010 Exam #1 Name: M^2

For full credit you must show sufficient work that the method of obtaining your answer is clear.

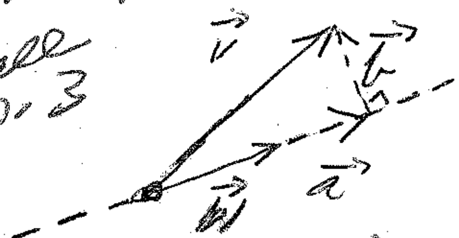
1. (30 points) Let P be the point $(-2, 1, 6)$; let $\mathbf{v} = \langle 4, -1, -5\rangle = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{w} = \langle 2, 1, -1\rangle$.

   a. Find the terminal point Q of $\mathbf{v}$ if the initial point is P.

$$(-2+4,\ 1+-1,\ 6-5) = (2, 0, 1)$$

typo! (reversed) see p. 3

   b. Compute the orthogonal projection of $\mathbf{v}$ on $\mathbf{w}$ (that is, $\mathbf{a} = \text{proj}_{\mathbf{w}}\mathbf{v}$, ~~the component of v in the~~ direction of $\mathbf{w}$). Then compute the component of $\mathbf{w}$ that is orthogonal to $\mathbf{v}$; call this vector $\mathbf{b}$. Finally, verify that $\mathbf{a}$ is orthogonal to $\mathbf{b}$.



$$\vec{a} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{8-1+5}{4+1+1}\langle 2,1,-1\rangle$$

$$= \frac{12}{6}\langle 2,1,-1\rangle = \langle 4,2,-2\rangle$$

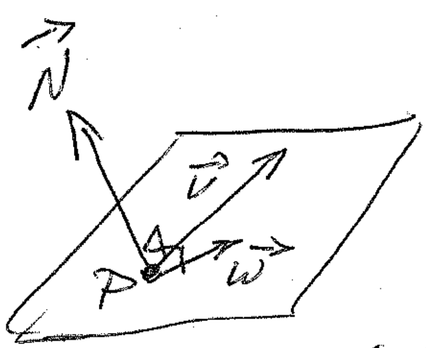
$$\vec{b} = \vec{v} - \vec{a} = \langle 0, -3, -3\rangle$$

$$\vec{a}\cdot\vec{b} = 0-6+6 = 0 \quad \text{so } \vec{a} \perp \vec{b}$$

   c. Give an equation for a plane whose normal vector is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and which contains the point P.

Take $\vec{N} = \vec{v}\times\vec{w} = \langle 1-(-5), -10-(-4),$

$4-(-2)\rangle = \langle 6,-6,6\rangle$



Check $\vec{N}\cdot\vec{v} = 24+6-30 = 0$

$\vec{N}\cdot\vec{w} = 12-6-6 = 0.$

$$6(x+2) - 6(y-1) + 6(z-6) = 0$$

$$6x - 6y + 6z = 18$$

or $\quad x - y + z = 3$

2. (20 points) A line L_1 passes through the points $(-3,-1,3)$ and $(1,1,-1)$. A line L_2 has equations $\frac{x+3}{-1} = \frac{y-4}{2} = \frac{z-3}{1}$. ← put $= s$

a. Get a direction vector $\mathbf{v}$ for L_1 and then write the parametric equations for this line. Give a third point that is on L_1.

$$\vec{v} = \langle 1-(-3),\ 1-(-1),\ -1-3 \rangle = \langle 4, 2, -4 \rangle$$

$$L_1 \begin{cases} x = -3+4t \\ y = -1+2t \\ z = 3-4t \end{cases}$$

Put $t=2$ for example $(5, 3, -5)$.

b. Is the point $(-2,2,4)$ on L_2? How can you tell?

$$\frac{-2+3}{-1} \stackrel{?}{=} \frac{2-4}{2} \stackrel{?}{=} \frac{4-3}{1}$$

$$-1 \stackrel{\checkmark}{=} -1 \stackrel{No}{=} 1$$

c. These two lines DO intersect. Find the point of intersection.

Method I Put the eq'ns for L_1 into those for L_2 and if we get a consistent t.

$$\frac{(-3+4t)+3}{-1} = \frac{(-1+2t)-4}{2} = \frac{(3-4t)-3}{1}$$

$$-4t = \frac{-5+2t}{2} \qquad \frac{-5+2t}{2} = -4t$$

$$-8t = -5+2t$$

same algebra!

$$-10t = -5$$

$$t = \frac{1}{2} \qquad t = \frac{1}{2}$$

So the point is $(-1, 0, 1)$

Method II Write param eq'ns for L_2 using a new variable.

$$\begin{cases} x = -3-s \\ y = 4+2s \\ z = 3+s \end{cases}$$

Then solve for t and s

$$-3+4t = -3-s \quad ① \quad x=x$$
$$-1+2t = 4+2s \quad ② \quad y=y$$
$$3-4t = 3+s \quad ③ \quad z=z$$

Notice ③ is just -1 times ①.
From ① $s=-4t$. Then from
② $-1+2t = 4+2(-4t)$

$$10t = 5$$
$$t = \tfrac{1}{2} \quad (x,y,z)=(-1,0,1)$$

$$s=-2 \quad (x,y,z)=(-1,0,1)$$

#1b I apologize for this typo. If you went on auto-pilot like I did (along with the previous quiz), you'll get full credit. If you actually did what I wrote then

$$\vec{b} = \vec{w} - \text{proj}_{\vec{v}}\vec{w} = \vec{w} - \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

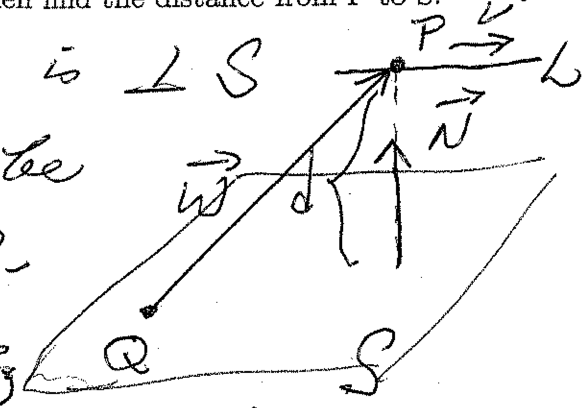
$$= \vec{w} - \frac{12}{16+1+25}\vec{v} = \langle 2,1,-1\rangle - \frac{\cancel{12}\,2}{\cancel{42}\,7}\langle 4,-1,5\rangle$$

$$= \left\langle \frac{6}{7}, \frac{9}{7}, \frac{3}{7}\right\rangle \text{ and } \vec{a}\cdot\vec{b} = \frac{36}{7} \neq 0$$

So $\vec{a}$ is NOT perp. to $\vec{b}$.

3. (20 points) Give equations for a line L through the point P $(1, 0, -2)$ that is **parallel** to the plane S: $3x - y + 2z = 5$. (a) (b) Then find the distance from P to S.

(a) $\vec{N} = \langle 3, -1, 2 \rangle$ is $\perp S$



Want $\vec{v}$ for L to be $\perp \vec{N}$, so $\vec{v} \cdot \vec{N} = 0$.

$\vec{v} = \langle 1, 1, -1 \rangle$ works, since $\vec{v} \cdot \vec{N} = 0$. So does $\vec{v} = \langle 0, 2, 1 \rangle$ and lots more.

$$L: \begin{cases} x = 1 + t \\ y = 0 + t \\ z = -2 - t \end{cases}$$

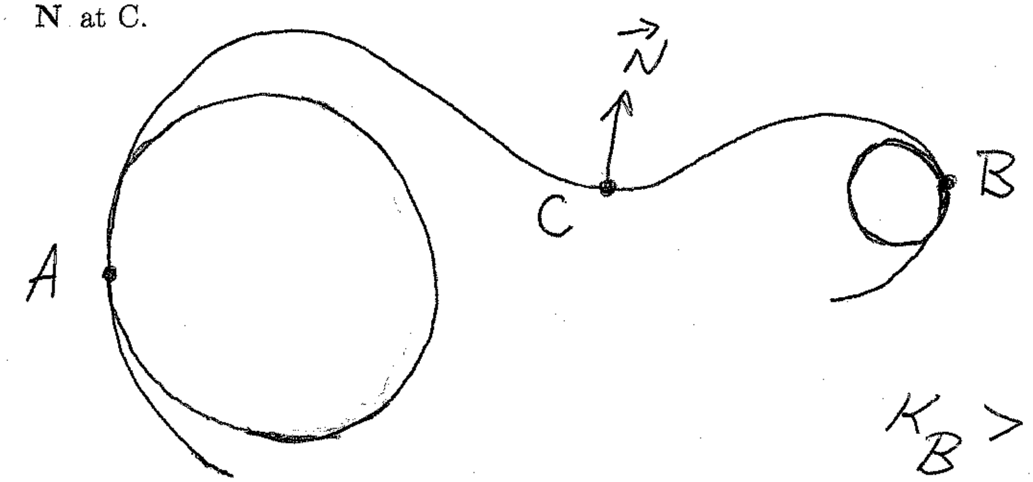
(b) Take any Q on S, say $(1, 0, 1)$. Then $d = \|\vec{a}\|$ where $\vec{a} = \text{proj}_{\vec{N}} \vec{w}$

$$\vec{w} = \overrightarrow{QP} = \langle 0, 0, -3 \rangle$$

$$\vec{a} = \frac{\vec{w} \cdot \vec{N}}{\vec{N} \cdot \vec{N}} \vec{N} = \frac{-6}{9+1+4} \vec{N} = \frac{-3}{7} \vec{N}$$

$$\|\vec{a}\| = \frac{3}{7} \|\vec{N}\| = \frac{3}{7}\sqrt{14}$$

4. (5 points) On the curve illustrated below, let κ_A and κ_B represent the curvatures at the points A and B, and let ρ_A and ρ_B represent the respective radii of curvature. Where is the curvature larger? Where is the radius of curvature larger? Show the osculating circles. Also show the unit normal vector **N** at C.



$\kappa_B > \kappa_A$

$\rho_A > \rho_B$

5. (25 points) A particle moves so that $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$ for $0 \le t \le 2\pi$.

a. Compute the velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$, the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t)$, the speed $\frac{ds}{dt} = ||\mathbf{v}(t)||$, and the unit tangent vector $\mathbf{T}(t)$.

$$\vec{v}(t) = \langle -2\sin t, 2\cos t, 1\rangle$$

$$\vec{a}(t) = \langle -2\cos t, -2\sin t, 0\rangle$$

$$\frac{ds}{dt} = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5} \quad \underline{\underline{\text{constant!}}}$$

$$\vec{T}(t) = \frac{1}{||\vec{r}'(t)||}\vec{r}'(t) = \frac{1}{\sqrt{5}}\langle -2\sin t, 2\cos t, 1\rangle$$

b. Compute the arclength s of the path from time $\tau = 0$ to $\tau = t$. How far has the particle traveled over the whole interval from 0 to 2π?

$$s = \int_0^t ||\vec{r}'(\tau)||\,d\tau = \int_0^t \sqrt{5}\,d\tau = \sqrt{5}\,t$$

whole distance is $\sqrt{5}(2\pi)$.

c. (Bonus) Recall that $\mathbf{a}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$, where $a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$, $a_N = \kappa\left(\frac{ds}{dt}\right)^2$. Compute the curvature κ from one of the formulas $||\mathbf{T}'(t)||/||\mathbf{r}'(t)||$, or $||\mathbf{r}'(t)\times\mathbf{r}''(t)||/||\mathbf{r}'(t)||^3$, or by using $||\mathbf{a}(t)||^2 = a_T^2 + a_N^2$.

$$||\vec{a}(t)||^2 = 4\cos^2 t + 4\sin^2 t = 4$$

$$a_T = \frac{d}{dt}(\sqrt{5}) = 0$$

$$a_N^2 = \kappa^2\left(\frac{ds}{dt}\right)^4 = \kappa^2(25) = ||\vec{a}||^2 - a_T^2 = 4$$

$$25\kappa^2 = a_N^2 = 4$$

$$\kappa^2 = \frac{4}{25} \qquad \kappa = \frac{2}{5}$$