**MATH 241 Spring, 1996 Final Exam Name:____________________**

*Show your work for full credit. Calculators are allowed. There are 167 points.*

1. (12 points) Let $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, and $P$ be the point $(-1, 4, -2)$.

   a. If $\mathbf{w} = \overrightarrow{PQ}$, compute the coordinates of the point $Q$.

   b. Compute the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   c. Compute $\mathbf{a} = \mathrm{pr}_{\mathbf{w}}\mathbf{v}$, the vector projection of $\mathbf{v}$ along $\mathbf{w}$.

   d. Explain why the vector $\mathbf{v} - \mathbf{a}$ is perpendicular to $\mathbf{w}$

2. (8 points) A line $\ell_1$ has symmetric equations $\frac{x}{2} = \frac{y-1}{-3} = \frac{z+4}{1}$, and a line $\ell_2$ has parametric equations $x = 3t + 1$, $y = -4t - 1$, $z = t - 3$.
   a. Find an equation for the plane that contains the point $(4, -1, -3)$ and the line $\ell_1$.

   b. Do the two lines intersect? If yes, find the point of intersection; if no, explain why not.

3. (8 points) A particle moves with velocity $\mathbf{v}(t) = (e^t, 2t)$. At $t = 0$ the position is $(2, -1)$.
   a. Compute the position vector $\mathbf{r}(t)$.

   b. Set up, but do NOT compute, the integral that gives the distance traveled from $t = 0$ to $t = 3$.

4. (8 points) Let $f(x, y, z) = x^2y + z^2x^3$ .
   a. Compute $\operatorname{grad} f$ at the point $Q(-2, 3, 1)$ .

   b. What is the maximum value of any directional derivative of $f$ at $Q$ ?

5. (6 points) The formula $P = 40L^{1/4}K^{3/4}$ gives the production of a factory in terms of the amounts of labor $L$ and capital invested $K$ (in suitable units). *Use differentials, or the microscope approximation,* to determine the *approximate* change in $P$ if $K$ changes from 16 to 17.2 and $L$ changes from 81 to 80.1.

6. (6 points) Sketch the portion of the surface $3x + 12y + 8z = 24$ that lies in the first octant. Then *set up, but do NOT compute* a double or triple integral that gives the volume of the solid in the first octant bounded by the surface and the coordinate planes.

7. (10 points) Match the contour diagrams with the surfaces. In each case, assume that adjacent contours in the diagram go with equal changes in the value of $z$ .
I- , II- , III- , IV- , V- .

8. (6 points) Let $A = \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$ . Show how $A$ transforms the unit square $\{(x, y) \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ . Find the area of the transformed region and justify your answer.

9. (6 points) The contour diagram shown below gives the weekly revenue $R$ of a music store as a function of $T$, the number of tapes sold, and $C$, the number of compact discs sold.

   a. Explain why the diagram shows that $R$ is a linear function of $T$ and $C$.

   b. Determine the unit price of a tape and a disc.

10. (12 points) Let $h(x,y) = 3y^3 + 6xy - x^2$. Find all critical points for $h$ and indicate–with explanation–whether each is a local max, local min, or saddle point.

11. (8 points)

a. A vector field $\mathbf{G}$ is shown below. Determine if $\mathbf{G}$ is conservative or not, basing your argument on line integrals, NOT on a formula for $\mathbf{G}$.

b. A vector field $\operatorname{grad} f$ is shown below. Sketch contour lines for the function $f(x, y)$. Indicate the places on the diagram where $f$ most likely has a local max, a local min, or a saddle point, and explain how you know.

12. (9 points) Let $\mathbf{F} = e^{-x}\cos y\mathbf{i} + e^{y}\sin x\mathbf{j} + z\mathbf{k}$.

a. Compute $\operatorname{div}(\mathbf{F})$.

b. Compute $\operatorname{curl}(\mathbf{F})$

c. Is $\mathbf{F}$ conservative or not? Explain!

13. (18 points) Compute each iterated integral. Change the order of integration, or change the coordinate system, if necessary.

a. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1}^{4-x^2-y^2} \frac{1}{z^2}\, dz\, dy\, dx$

b. $\int_{0}^{6} \int_{x/3}^{2} x\sqrt{y^3+1}\, dy\, dx$

14. (16 points) Let $w = g(x,y,z) = x^2 + y^2 - 4z^2$ .

a. Sketch the level surfaces (and clearly label which is which!) $w = -36$, $w = 0$, and $w = 16$.

b. The point $P(1, -1, 1/2)$ is on what level surface for $w$? Give an equation for the tangent plane to that surface at the point $P$.

15. (22 points) Compute each line integral.

a. $\oint_C x^2\,dx + xy\,dy$ , where $C$ is the boundary of the triangle PQR, oriented counterclockwise, with vertices $P(0,0)$ , $Q(3,0)$ , and $R(3,1)$ , by explicit calculation of the integral along the three edges.

b. the same integral as in a., but by using Green's Theorem

c. $\int_C \mathbf{G} \cdot d\mathbf{r}$ , where $\mathbf{G} = (\frac{2xy}{1+x^2}, \ln(1+x^2))$ and $C$ is the line segment from $(0,3)$ to $(1,1)$ . Hint: find a potential function for the vector field $\mathbf{G}$ .

16. (12 points) *Set up, but do NOT compute* each of the following integrals.

a. the triple integral, in spherical coordinates, for the volume of the region inside the sphere $x^2+y^2+z^2 = 9$ and outside the double cone $z^2 = x^2+y^2$ .

b. the volume bored out of the sphere $x^2+y^2+z^2 = 36$ by a circular cylinder of radius 3 whose surface contains the $z$-axis (the cylinder is "off center").

That's it, folks! Enjoy your summer vacation!