Fall, 2011 **Worksheet 7** **Math 172**

**Systems with several dependent variables,
equilibria and stability**

1. Read Gotelli, chapter 6, and do problem 6.1.. You can beef up problem 6.1 by finding the equilibrium, and determining the population trajectories in the phase portrait for a variety of different initial conditions (making the diagram of arrows can help). We will also review how to convert this continuous model into a discrete one by using Euler's method, and thereby get our calculator to make these plots (suggestion: take $\Delta t = 0.2$ and go for at least 10 time units, in which case $n = 0$ to what?).

2. A mild infectious illness, spread by close contact, is making people ill (and infectious). Each week there is a mass action interaction with transmission coefficient 0.47 of the healthy, but susceptible, population $S(t)$ with the infectious population $I(t)$. At the same time 75% of the sick and infectious population recovers and is immune to getting re-infected; the recovered population is denoted $R(t)$. The time span is short enough that no one dies in the period under observation. As the number of sick people increases, they are spreading out over a larger area and more healthy people are put into a situation in which they might become infected; in fact the pool of susceptible people increases by 10 thousand people per week. Write a system of continuous model equations for for this process (the unit of measurement is thousands of people per week). Besides being a continuous model, this differs from the previous one in having a mass action interaction term–be careful!

3. A not very infectious illness, but one that is occasionally fatal, is spreading through a susceptible population $S(t)$. Each week there is a mass action interaction with transmission coefficient 0.04 of the susceptible population $S(t)$ with the ill and infectious population $I(t)$ in which susceptible individuals become ill and infectious. At the same time 33% of the infectious population recovers; the recovered population is denoted $R(t)$. Just 0.5% of the infected population dies. Most of the recovered population, in fact 92%, is immune to reinfection, but the rest do become susceptible again. Write a system of **continuous** model equations for for this process (the unit of measurement is thousands of people per week).

4. A matrix M has eigenvectors $\mathbf{e}_1 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. These go with eigenvalues $\lambda_1 = 1.02$ and $\lambda_2 = 0.95$, respectively. We have $\mathbf{u}_0 = \mathbf{e}_1 + 2\mathbf{e}_2$.
   a. Compute $\mathbf{u}_1 = M\mathbf{u}_0$. You may leave the symbols $\mathbf{e}_1$ and $\mathbf{e}_2$ in your answer.

b. Compute $M^2\mathbf{u}_0$. You may leave the symbols $\mathbf{e}_1$ and $\mathbf{e}_2$ in your answer. This is a computation of $\mathbf{u}_k$ for which value of k?

c. Rewrite $M^n\mathbf{u}_0$ in the form $a\,\mathbf{e}_1 + b\,\mathbf{e}_2$. This computation is used to compute what quantity?. What happens to a and to b as n gets larger and larger?

d. If M is a population projection matrix (Leslie matrix), what is the long term growth rate of the population? What is the stable age distribution?

5. Consider the following continuous model of a predator-prey system.

$$\frac{dV}{dt} = 0.5V(1-\frac{V}{250}) - 0.02VP = V\left(0.5(1-\frac{V}{250}) - 0.02P\right)$$
$$\frac{dP}{dt} = -0.8P + 0.004VP = P(-0.8 + 0.004V)$$

a. What kind of growth does the victim population exhibit if there are no predators (i.e., $P = 0$)? Why is $(V^*, P^*) = (250, 0)$ an equilibrium, and how do you interpret this biologically?

b. Compute the equilibrium (V^*, P^*) other than $(0,0)$ and $(250,0)$ for this system.

c. Mark the coordinates of the equilibrium pts (heavy dots), and place the predator population arrows (up or down), victim population arrows (left or right), and net population change arrows at the open dots.

6. A predator exhibits a type II (Michaelis-Menton) functional response to prey abundance V (measured in thousands) by having a per capita kill rate $R = \dfrac{12V}{4+V}$.

a. What is a good approximation of R if V is very small?

b. What is a good approximation of R if V is very large?

c. At what value of V is R one half of the maximum kill rate? What is the maximum kill rate?

d. Plot R as a function of V, exhibitng the features of a., b., c.

7. Rewrite the discrete dynamic model equation $w_{n+1} = -0.2w_n + 24$ in calculator-ready form, that is, give $w(n)$ in terms of $w(n-1)$.

a. Suppose $w_0 = 40$. What happens to w_n in the short term? (The calculator might be best for this.)

b. What happens to w_n as $n \to \infty$? Does w_n increase, decrease, oscillate, tend towards or away from the equilibrium? What is the equilibrium value and is it stable or not? Explain. (The calculator may give you an idea, but the explicit solution will absolutely nail it down.)