Fall, 2011 **Worksheet 5** **Math 172**

Matrices, vectors, stationary distributions, eigenvalues and eigenvectors

Read Gotelli, chapter 3, especially pages 59–66 and 71–79. and chapter 8, pages 180-199. We explore Leslie models, then the somewhat more general Lefkowitch models, and finally models of succession. All use pretty much the same ideas. There is a transition matrix A, an initial vector $\mathbf{P}_0$ and the relationship $\mathbf{P}_{t+1} = A\mathbf{P}_t$. We are interested in the long term behavior, especially in situations in which we can detect a "stable age distribution" or something similar. The big differences between Leslie-Lefkowitch models and successional models are that in the latter all entries of the transition matrix are probabilities so that the columns add up to 1, and instead of reproduction we have the possibility of regression from a "later" stage to an "earlier" one. We have a perfect example of this process just down the road at the Congaree Swamp National Park: after Hurricane Hugo tore through, large holes were made in the forest canopy and at ground level by the uprooted trees. Fast growing sunlight loving plant species now had an opportunity to flourish,but gradually gave way to slower growing species that could tolerate shade (read about the facilitation and inhibition models of succession in Gotelli).

1. Let vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and matrix $A = \begin{bmatrix} 1.3 & -0.2 \\ 0.15 & 0.9 \end{bmatrix}$.
   - a. Compute $\mathbf{u} = 3\mathbf{v} + \mathbf{w}$, $A\mathbf{u}$, and $3A\mathbf{v} + A\mathbf{w}$. What do you notice?
   - b. Compute A^2.
   - c. Plot $\mathbf{v}$ and $A\mathbf{v}$; plot $\mathbf{w}$ and $A\mathbf{w}$. The eigenvalues for this matrix are $\lambda = 1.2$ and $\mu = 1$, and eigenvectors are $\mathbf{v}$ and $\mathbf{w}$. Which goes with which?
   - d. One of the given vectors represents a steady state. Find a vector that lines up with this one, but whose column total is 1 (this is called *normalization*).

2. Given below is the transition matrix A for a weather model with three states: sunny (S), cloudy (C) and rainy (R).

$$\begin{array}{r} \text{tomorrow}\downarrow \backslash \text{today}\rightarrow \\ S \\ C \\ R \end{array} \quad \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 1/2 & 0 & 1/8 \\ 1/4 & 1/4 & 1/8 \\ 1/4 & 3/4 & 3/4 \end{bmatrix} \end{array}$$

   - a. If it is rainy today, what is the probability that it is cloudy tomorrow?
   - b. If it is rainy today, what is the probability that it is cloudy the day after tomorrow? (Hint: don't do more work that is absolutely necessary!)
   - c. If it is rainy today, what is the probability that it is cloudy 10 days from now?
   - d. What is the significance of the vector $\mathbf{v} = \begin{bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{bmatrix}$?

3. You are given annual survival probabilities and fecundities (fertilities) for a population of frogs with three stages: tadpoles given by T_n, juveniles given by J_n, and adult frogs given by F_n. Produce the transition matrix A if tadpoles have a 20% chance of becoming juveniles, juveniles have a 10% chance of remaining juveniles and a 40% chance of becoming grown up frogs, and adults have a 55% chance of survival (as adults, of course). On average each juvenile produces 2 surviving tadpoles per year, and each adult produces 80.
   a. Construct the transition matrix A based on the figures given below from one census to the next. Notice this is a stage-based rather than age-based model.
   b. If there are currently 100 tadpoles, 20 juveniles, and 20 adult frogs, what is the current distribution? Compute the population and distribution vectors for each of the next two years.
   c. What is (observationally) the long term behavior of this system?

4. This is the problem that we began in class. There are 4 age classes, with populations H_t, J_t, S_t, and A_t. We set $\mathbf{P}_t = \begin{bmatrix} H_t \\ J_t \\ S_t \\ A_t \end{bmatrix}$ and are given a transition matrix $M = \begin{bmatrix} 0 & 0 & 2 & 6 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \end{bmatrix}$.
   a. What is the mortality rate of the juveniles? With what probability does a juvenile grow up to become an adult in 2 years?
   b. In class we used $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 10 \\ 10 \\ 10 \end{bmatrix}$, and we computed $\mathbf{P}_t$ for $t = 1, 2, 10, 11$ using $\mathbf{P}_{t+1} = M\mathbf{P}_t$ and $\mathbf{P}_t = M^t\mathbf{P}_0$. We also found the corresponding distribution vectors $\mathbf{D}_t$, by totaling up the entries of $\mathbf{P}_t$ and dividing through by this number. Continue this process for $t = 20, 21, 25, 26$. What do you observe about the $\mathbf{D}_t$ vectors? How are the $\mathbf{P}_t$ vectors growing when t is large? Suggestion: compute the ratio of the total population at $t+1$ to that at t; do the same for each age group as well. How would you describe the long term annual growth rate of this population?
   c. Suppose now we change the initial population to be $\mathbf{P}_0 = \begin{bmatrix} 30 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
   Do the final computations of part (b) over again. Which answers are different,which are the same, and what conclusion do you draw from this?

5. We will now modify the model above to construct a stage-based model. The

transition matrix in this case is called a Lefkowitch matrix. You will see many of these illustrated in the text. The hatchling stage is still just one year,with a survival probability to the juvenile stage of 0.3. The juvenile stage still has an annual survival probability of 0.7 in all, but just 0.4 probability of moving on to the sub-adult stage, and 0.3 probability of surviving, but remaining in the juvenile stage. The subadults still produce 2 surviving hatchlings per individual per year, and still have a total 0.9 probability of survival, but this is made up of 0.2 probability of remaining a sub-adult, and 0.7 probability of moving up into the adult stage. Now comes the real change: adults still produce 6 surviving hatchlings per individual per year, but have a survival rate of 50% as adults.

a. What is the mortality rate of each stage? What is the probability that an adult will survive as an adult for 2 years? for 3 years?

b. Write down the new transition matrix M and compute M^2.

c. Use the first $\mathbf{P}_0$ of the previous problem. Compute $\mathbf{P}_n$ and the corresponding $\mathbf{D}_n$ for $n = 1, 2, 10, 11, 20, 21, 25, 26$. Compare the numbers to the ones that you obtained above; what do you observe? What do you observe about the $\mathbf{D}_t$ vectors? How are the $\mathbf{P}_t$ vectors growing? Suggestion: compute the ratio of the total population at $t+1$ to that at t; do the same for each age group as well. How would you describe the long term annual growth rate of this population?

d. The dominant eigenvalue for M is $\lambda = 1.171$, which goes with an eigenvector $\mathbf{v} = \begin{bmatrix} 3356 \\ 1156 \\ 476 \\ 496 \end{bmatrix}$. We find that $\mathbf{P}_{21} = \begin{bmatrix} 1776 \\ 612 \\ 252 \\ 263 \end{bmatrix}$. Has the population reached its stable stage distribution at $t = 21$? Explain. Use λ to get an estimate for $\mathbf{P}_{23}$. Explain why the same method cannot be used to get $\mathbf{P}_2$ from $\mathbf{P}_0$.

6. A matrix A has eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. These go with eigenvalues $\lambda_1 = 0.9$ and $\lambda_2 = -1.2$, respectively. We have $\mathbf{P}_0 = 6\mathbf{v}_1 - \mathbf{v}_2$.

a. Compute $\mathbf{P}_1 = A\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

b. Compute $\mathbf{P}_2 = A^2\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

7. A matrix M has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. These go with eigenvalues $\lambda_1 = 1.04$ and $\lambda_2 = 0.8$, respectively. We have $\mathbf{u}_0 = \mathbf{v}_1 + 2\mathbf{v}_2$.

a. Compute $M(\mathbf{u}_0)$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

b. Compute $M^2(\mathbf{u}_0)$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer. This is a computation of $\mathbf{u}_k$ for which value of k?

c. Rewrite $M^n(\mathbf{u}_0)$ in the form $a\,\mathbf{v}_1 + b\,\mathbf{v}_2$. This is a computation of what quantity? What happens to a and to b as n gets larger and larger?

d. If M is a population projection matrix (Leslie matrix), what is the growth rate of the population? What is the stable age distribution?

8. You are given annual survival probabilities and fecundities (numbers of surviving offspring) for a population with three stages: hatchlings H_t, juveniles J_t, and adults A_t. Hatchlings have a 10% chance of becoming juveniles; juveniles have an 80% chance of remaining juveniles and a 5% chance of becoming adults. Juveniles produce on average two hatchlings for the next year. Adults have a 90% chance of remaining alive as adults, and produce 6 hatchlings on average for the next year.

   a. Construct the transition matrix M.

   b. The initial population vector is $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$; compute $\mathbf{P}_1$ and $\mathbf{P}_2$.

   c. Verify that $\mathbf{P}_{31} = \begin{bmatrix} 368 \\ 119 \\ 28 \end{bmatrix}$. What is the distribution of the population into the various stages after 31 years? Hint: How is $\mathbf{P}_{31}$ computed in terms of M and $\mathbf{P}_0$?

   d. Compute $\mathbf{P}_{32}$, and compare the distributions at times 31 and 32. What do you observe?

   e. You also find that M has three eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$, and that $\mathbf{P}_0$ can be written in terms of these.

$$\lambda_1 = -0.176 \qquad \lambda_2 = 0.767 \qquad \lambda_3 = 1.109$$

$$\mathbf{v}_1 = \begin{bmatrix} 6.629 \\ -0.679 \\ 0.032 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0.183 \\ -0.551 \\ 0.207 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1.194 \\ 0.386 \\ 0.092 \end{bmatrix}$$

$$\mathbf{P}_0 = 13.071\mathbf{v}_1 - 7.472\mathbf{v}_2 + 12.326\mathbf{v}_3$$

   f. Get an exact formula for $\mathbf{P}_t$. You may leave the $\mathbf{v}_i$'s in your answer.

   g. Give a formula for $\mathbf{P}_t$ that approximates it well when t is large.

   h. What is the approximate annual growth rate?

   i. In the long run what growth stage forms the majority of the population, and how can you tell?

9. In a 2-variable system you find that eventually $u_n \approx 1.02u_{n-1}$ and $v_n \approx 1.08v_{n-1}$. Is the total population $T_n = u_n + v_n$ growing or declining? Someone claims that the total population must be growing at $(2+8)/2 = 5\,\%$ a year–is this correct? Is there eventually a stable distribution in which each of u_n and v_n maintains less than 100% of the population? Briefly explain your answers.

10. Use the matrix at the bottom of page 197 in the text. Can you replicate the discussion of Figures 8.6 and 8.7?