Math 172 WS 4 — Solving model equations — Fall, 2011

From the purely mathematical point of view we are beginning to solve differential equations. At first we will be successful in finding formulas (explicit solutions), but eventually we will see that the real world is not so tidy, and we will have to develop numerical, computational methods. Somewhere in between, and helpful to both approaches, is to use intuitive graphical observations that come from the differential equation or model equation itself. For discrete models equations (and continuous ones made discrete by using Euler's method), we can use the calculator.

In the next few problems we consider the discrete affine mode $Q_{t+1} = aQ_t + b$. Typically a has the form $1+r$, but you have to read carefully how the model is described.

1. A quantity Q_t satisfies the affine discrete model equation $Q_{t+1} = 0.4Q_t - 48$.
   a. Find the equilibrium value Q^* (this is just doing the math, so don't worry if it seems unrealistic biologically). By replacing Q_t by $P_t = Q_t - Q^*$, show that P_{t+1} satisfies a much simpler model, namely $P_{t+1} = 0.40P_t$; in fact you can solve this equation! Now reverse the process, replacing the P's by Q's as appropriate to conclude that $Q_t = C(0.4)^t + Q^*$.
   b. Notice that if $a = 0.4 = 1 + r$, then r is what? If $Q_0 = -120$, what is C? What is the long term behavior of Q_t (as $t \to \infty$)?

2. Check that $P_t = C(1+r)^t + P^*$ does in fact satisfy the model equation $\Delta P = P_{t+1} - P_t = rP_t + m$ or $P_{t+1} = (1+r)P_t + m$, where P^* is the equilibrium value and C is a constant.
   a. Show that $P^* = -m/r$. How would you determine C?
   b. Taking the above for granted, what does all this boil down to if $m = 0$?
   c. Explain why m and r should have opposite signs for P^* to make biological sense.
   d. To use this formula effectively we need to know how $a^t = (1+r)^t$ behaves as $t \to \infty$. This turns out to be more complicated that knowing how e^{rt} behaves in the continuous case. Describe how $(1+r)^t$ behaves as $t \to \infty$ in each of these cases: $r = 0.2$, $r = -0.2$, $r = -1.2$, $r = -2.2$.

3. The amount of drug in the bloodstream (measured in mg) is given by $u_n = 0.6u_{n-1} + 200$, where n is in days. Notice we have written this model in "calculator-ready" language.
   a. What term in this model represents the daily dose, and which term represents the amount left from the day before?
   b. Suppose this dosing pattern started at a time when there was 100 mg of the drug in the bloodstream. Compute u_n in terms of n in this case. (Hint: recall that the model $u_n = au_{n-1} + b$ has solution of form $u_n = Ca^n + u^*$, where u^* is the equilibrium value, and C can be determined by using a specific u_n value.)
   c. What happens to u_n in the long term as this dosing continues? Explain.

4. We are given a discrete model $P_{n+1} = (-0.9)P_n + 95$ with $P_0 = 70$.
   a. Find the explicit solution for P_n.
   b. What happens to P_n as $n \to \infty$? Does it increase, decrease, oscillate, tend towards or away from the equilibrium? Conclude whether the equilibrium is stable or not.

5. We are given a discrete model $S_{n+1} = (-1.3)S_n + 161$ with $S_0 = 100$.
   a. Find the explicit solution for S_n.
   b. What happens to S_n as $n \to \infty$? Does it increase, decrease, oscillate, tend towards or away from the equilibrium? Conclude whether the equilibrium is stable or not.

Next we consider the affine continuous mode, which is analogous to the discrete version.

6. Later we will consider harvesting models (these are very important in fisheries, for example). For now, let's just do some math. Suppose $\frac{dP}{dt} = 0.018P - 3.6$ and $P(0) = 50$ (the "initial condition").
   a. We know how to solve $\frac{dP}{dt} = P' = 0.018P$ alone and also how to solve $\frac{dP}{dt} = -3.6$ alone (right?), but try as we might there is no way to separate the variables. A little trick works, however. Notice that $\frac{dP}{dt} = 0$ when $P = 200$. This special value is written P^* (or sometimes $\hat{P}$). If we introduce a new variable U by the formula $U = P - P^* = P - 200$, show that $\frac{dU}{dt} = \frac{dP}{dt}$ and $U(0) = -150$. Turning the relationship around, we get $P = U + 200$. Use this along with the model equation for $\frac{dP}{dt}$ to compute $\frac{dU}{dt}$ in terms of U. This is an equation that you can solve, so do it. We are not interested in U, so rewrite your formula to get one for P in terms of t and constants. (I wouldn't ask you to do this on a test, so give it a try, but if you cannot get it, go on to part (b).)
   b. In the end you should end up with $P(t) = Ce^{0.018t} + P^*$, where C can be determined by using the initial condition. Graph the solution over time.
   c. If we now have the initial condition $P(0) = 300$, what do you now get for C and for the graph? Is this equilibrium stable or unstable?

If you did part (a) of the previous problem, or even if you did not, the upshot is that an affine continuous model $\frac{dQ}{dt} = Q' = aQ + b$ has an explicit solution $Q(t) = Ce^{at} + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition.

7. In a particular dessert habitat, a population of mice **declines** at a **per capita rate** of 3% yr $^{-1}$, but is reinforced by the in-migration of mice from destroyed nearby habitats at 24 mice/yr. Write a continuous model equation for this situation, and solve it, assuming that the initial mouse population is 1,000. What happens to the mouse population in the long term, and how do you know?

8. An invasive water weed is growing continuously in a local lake at an intrinsic rate of 3% a year. Lake management dredges up 60 tons of the weed each year. If $M(t)$ is the mass of the weed measured in tons, and t is measured in years, write the model equation for $M'(t)$. Determine if there is an equilibrium, and if so, compute it. Compute the explicit solution if the invasion was first detected when the weed mass was estimated to be 500 tons. Use your solution to determine if the weed grows out of control, if it approaches the equilibrium value, or if the harvesting eventually brings about elimination of the weed. In the first case, determine when $M(t) = 1500$; in the third case determine when the weed is completely eliminated.

9. Do problem 2.1 in Gotelli. Use the variable P for population.

10. Do problem 2.2 in Gotelli. It is very useful to make the graph of P' vs. P in this case to find the maximum population net growth rate at a particular value of P.

11. We can make use of problem 2.3 in Gotelli in the following way. Set $\frac{dN}{dt} = g(N)N$, with per capita growth rate $g(N) = b' - d'$ as given in the text. Combine terms and factor out -0.0005 (which is actually -1/2000); you should get $g(N) = (-1/2000)(N^2 - 40N + 200)$. The roots of this polynomial are approximately $c_1 = 34$ and $c_2 = 6$ (you can use your calculator to find them more exactly). So we actually get a factorization $g(N) = (-1/2000)(N - c_1)(N - c_2)$, or $g(N) = (1/2000)(N - c_1)(c_2 - N)$. Use this to analyze the original net growth rate $\frac{dN}{dt} = g(N)N$ by findiung equilibrium values, and determining their stability. What kind of model is this?

For the next two problems we abandon hope of an explicit solution, and just try to reason from the model equation itself, following the sign of the derivative as a cue. This gives us "qualitative" information very quickly.

12. Squid are commercially important for human consumption, especially around the Mediterranean and in SE Asia; they also play a large role in many marine food chains. Suppose that a local squid population, measured in tons, is controlled by the continuous dynamic model $\frac{dS}{dt} = 0.0007S(100 - S)(S - 10)$ tons/year. It is easy to separate the variables of this model equation, but hard to do the integration. Compute the equilibrium values for S. By carefully selecting different initial values, determine whether each equilibrium value is stable or unstable.

13. A fish population $F(t)$, measured in thousands, is controlled by the continuous model $\frac{dF}{dt} = 0.0005F(120-F)(F-20)$ thousand fish/year. It is easy to separate the variables of this model equation, but hard to do the integration. Find the equilibrium values of F, and examine the long term behavior of $F(t)$ subject to various initial conditions for $F(0)$. Which are stable, which are unstable? How would you describe the various values from a biological point of view?

These next few problems involve separation of variables to do the integration.

14. Find the solution of the model equation $\frac{dP}{dt} = g(t)P$, where the per capita growth rate (representing habitat quality over time) declines linearly: $g(t) = -\frac{r}{a}t + r$. Take $r = 0.014$ and $a = 28$. Assume that $P(0) = 1000$. Graph the solution.

15. If $\frac{dS}{dt} = 0.9$ and $S(0) = 23$, determine an explicit formula for $S(t)$. This one is easy!

16. If $\frac{dV}{dt} = 0.9t^2$ and $V(0) = 23$, determine an explicit formula for $V(t)$. This one is just a bit harder.