Math 172, Fall, 2011
Modeling Project I – Problem choice A

We shall model the oxygen concentration in the lung over the course of a single breath. As many quantities are changing simultaneously, and there are many intermediate steps that occur within a single breath, we will not attempt to model every aspect, but to track what happens to the concentration of oxygen in the lung, c_{n-1}, to see where c_n comes from. This is not hard, but demands some patience. In the end you should obtain a discrete affine equation amenable to calculator (appoximate) and hand (exact) solution.

First of all some notation. We will use V to denote the volume of the inflated lung, and W to represent the amount exhaled and then inhaled. As you all know W is generally quite a bit smaller than V; these quantities vary from person to person. The concentration of oxygen in the ambient (outside) air is 21. The main fact from physics or chemistry that you need to know is that concentration is amount of oxygen divided by volume, or amount of oxygen is volume times concentration. We shall assume that as soon as the outside air is breathed in, the old air and the new air mix up completely, giving a uniform concentration within the lung. Volume is measured in liters (ℓ), amount of oxygen in mmol, and concentration in mmol/ℓ.

Let us make up a table describing what happens as we breath. Fill it in line by line. Note that the total amount of oxygen in the lung at the end of the cycle comes from the remaining oxygen plus the inhaled oxygen. At this point we will ignore the crucial fact that oxygen is absorbed into the blood stream.

Action	Volume	Concentration	Amount of oxygen
Air in lung before breath	V	c_{n-1}	$c_{n-1}V$
Air exhaled from lung	W		
Air remaining in lung after exhaling	$V-W$		
Outside air inhaled	W		
Air in lung after breath	V	c_n	

Now we can write the final amount of air in the lung as a sum of two terms which involve c_{n-1}, V, W, and 0.21. If you divide throough by V you will see that V and W never appear separately, but only together as a ratio $q = W/V$, which we shall call q. This is the fraction of the lung volume that is exchanged, and varies from person to person. So we get c_n in temrs of c_{n-1}, q, and 0.21.

Now if we take $q = 0.2$, we can find the equilibrium concentration of the oxygen in the lung. Investigate what happens in the long term if the inital concentration starts above or below this amount. Investigate what happens if an athlete breathes; what do expect her value of q to be?

Assume that a fraction α of the oxygen in the lung is absorbed by the blood. Change the first line of the table to reflect the fact that the fraction α of c_{n-1} has been removed (or in other words, what fraction of c_{n-1} remains? Then rebuild the table to get a new model equation for c_n. Find the equilibrium for this model. Is it higher or lower than the value you found for the sitution when no oxygen was absorbed (just the simple in and out pumping of air from the lung)?

Using $q = 0.2$ and the ambient oxygen concentrationm of 0.21, it turns out that over time the actual oxygen concentration exhaled by the lung is 0.15. What value of α that causes this to happen? Now you are ready to try some different values of q and α; a typical value of α is 0.3. Investigate equilibria, long term behavior of c_n and stabilty.

Math 172, Fall, 2011
Modeling Project I – Problem choice B

We are going to develop two continuous models, one simple, and one more sophisticated, to describe the one step of the metabolism of alcohol in the body, by focusing on the amount in the bloodstream as one drinks, and the removal of this alcohol from the bloodstream by the liver. We are not going to worry about what happens in the liver, how the waste products are disposed, the effect on the brain cells, and so on. As this process is continuous, we will use continuous dynamic models. In the first model we assume that the quantity of alcohol in the bloodstream $Q(t)$, measured in grams, as a function of time t, measured in hours, is changing by consumption of c grams/hour and is being removed by the liver at a rate proportional to Q itself. Denote the constant of proportionality by r, and write the model equation that governs the changing Q. If you like you may use $L(t)$ to denote the amount of alcohol absorbed by the liver, and write the model equation for this as well, but we are not really interested in this part of the process, as I indicated above. In humans the value of r is around 2.5 hr^{-1} or 2.5/hr (this actually depends on weight, body composition (alcohol is absorbed into all tissues that contain water, not just the blood), and consequently, also gender). Since we are taking $r > 0$, be sure that your model equation(s) have the correct sign(s); remember that $Q(t)$ represents the amount of alcohol in the bloodstream at time t. Let us assume that $Q(0) = 0$, and for starters that $c = 14$ g/hr (this amounts to one "drink" such as a 12 oz. can of beer or a 5 oz. serving of wine or one oz. of 100 proof hard liquor per hour). Solve the model equation by obtaining an explicit formula for $Q(t)$ in terms of t. Is there an equilibrium value for Q? What happens to $Q(t)$ as time goes on? In this model, the liver continues to remove alcohol from the bloodstream no matter how much is consumed (how can you tell this from the model?). Unfortunately in real life, this does not happen; the liver can only take up so much so fast; in technical terms there is a level of "satiation" analogous to the satiation of a predator when prey are abundant; so we must modify the model to take this into account.

As usual we modify a simple model by changing a parameter, in this case r, into a variable quantity that better reflects the situation. We will use $g(Q) = 10/(4 + Q)$, which is approximately true for an "average" male; to avoid having more constants $a/(b + Q)$ floating around (to account for weight, body composition, gender, etc.), let's be content to use 10 and 4. Check that this function is compatible with our original model when $Q = 0$. As Q increases, what happens to $g(Q)$, and what does this signify about the liver's capacity to draw alcohol out of the bloodstream? A graph would be a good idea here. You should find the term $g(Q)Q = \frac{10}{4+Q}Q$ in your revised model; let us give this term the name $T(Q)$. Here T stands for the transfer function that represents the uptake of alcohol by the liver and simultaneous removal from the blood. Consider the behavior of this function $T = T(Q) = \frac{10}{4+Q}Q$. For Q very small in comparison to 4, we may as well simplify and say $4 + Q \approx 4$; how does this simplify $T(Q)$, and what would the graph of T as a function of Q look like? When Q gets up to 4, what is the value of T? As Q rises higher and higher above 4, we now have $4 + Q \approx Q$; what happens to T consequently? Put all this information together to graph the transfer function T as a function of Q. How does this reflect the fact that the liver can only take up so much alcohol per unit time, and cannot keep up with a rising level of Q? In biology this is called a type II functional response, and arises very often when a consumer is satiated (e.g., a predator in the presence of abundant prey eventually can consume no more in a certain period of time). The value 4 is called the "half saturation constant" – can you explain why?

Recap the revised model for $\frac{dQ}{dt}$ in terms of Q and constants (again be careful to use the appropriate signs). Does this revised model have an equilibrium value for Q? You may want to see how different values of c (representing consumption rate) affects the answer to this question. What happens if the value of Q is lower than, or, on the other hand, exceeds this equilibrium value, espcially in the long run if consumption continues at the same rate? You will probably have to use Euler's Method to approximate $Q(t)$ for specific values of t, as the rate equation (model equation) does not readily yield a formula solution. An hour by hour calculation will be too crude; you will need to use a Δt of maybe 15 minutes (0.25 hr) or even 6 minutes (0.1 hr) to get sufficient accuracy.

As a very rough conversion the blood alcohol content as measured by cops with a breathalyzer on a 140 pound male is BAC $= 0.002Q$, and the units come out so that this is a pure number that is treated as a percent. So, for example, $Q = 40$ grams gives the legal BAC cutoff of 0.08% (it is actually a pretty complicated process to explain percent of what, and has to do with the volume of blood in the body, the fraction of the blood that is water, the density of alcohol in water, etc.). Discuss how the equilibrium values that you found, and the long term behavior of $Q(t)$ will be reflected by the BAC.

(Bonus) In real life, drivers stop drinking at a certain point, so c drops down to zero. Let's say your 140 pound guy drinks continuously at a rate of 2 drinks an hour for three hours, stops, and wants to drive home. How long does he have to wait until he is legal? Your model(s) will have to include the $T(Q)$ term, and application of Euler's Method should use a small Δt.