MATH 172 **Fall, 2009** **Exam #3** **Name:** M^2

For full credit you must show sufficient work to justify your answer. Recall that the geometric series $\sum_{n=0}^{\infty} ar^n$ has a sum $S_\infty = a/(1-r)$ under a certain codition on r, which you should know, and fails to exist otherwise.

1. (10 pts) A reproductive female in the oldest stage of development produces 48 offspring on average each year. Her annual survival rate is 60%. What is her expected lifetime production of offspring?

$48 + 48(0.6) + 48(0.6)^2 + \cdots$
is a geometric series with $a = 48$, $r = 0.6$.
Now $|0.6| < 1$, so the sum is $S_\infty = \frac{a}{1-r} = \frac{48}{0.4}$
$= \boxed{120}$

2. (12 pts) Find the sum if it does exist, or state that there is no sum, and why.

a. $\sum_{n=1}^{\infty} \frac{5}{3}(-\frac{2}{3})^n = \underbrace{\sum_{n=0}^{\infty}(\frac{5}{3})(-\frac{2}{3})^n}_{} - \frac{5}{3} = \frac{5/3}{1-(-\frac{2}{3})} - \frac{5}{3}$

missing $n=0$ term

$a = \frac{5}{3}$ $r = -\frac{2}{3}$ full geom series $= 1 - \frac{5}{3} = \boxed{-\frac{2}{3}}$

$|-\frac{2}{3}| < 1$ OR Use $a = (\frac{5}{3})(\frac{-2}{3})$, $r = \frac{-2}{3}$.

b. $\sum_{j=0}^{\infty}(-\frac{2}{3})(\frac{5}{3})^j$

$a = -\frac{2}{3}$ $r = \frac{5}{3} > 1$ so there is $\boxed{\text{no sum}}$.

3. (10 pts) Compute the equilibrium point (u^*, v^*) of the discrete model system

$u^* = u_n = u_{n-1}$
$v^* = v_n = v_{n-1}$

$$u_n = 2u_{n-1} - 2v_{n-1} + 4$$
$$v_n = -3u_{n-1} + 4v_{n-1} + 9$$

$\begin{cases} u^* = 2u^* - 2v^* + 4 \\ v^* = -3u^* + 4v^* + 9 \end{cases}$

$\begin{cases} -4 = u^* - 2v^* \\ -9 = -3u^* + 3v^* \end{cases}$ $\frac{1}{3}$

$\begin{cases} -4 = u^* - 2v^* \\ -3 = -u^* + v^* \end{cases}$

add up

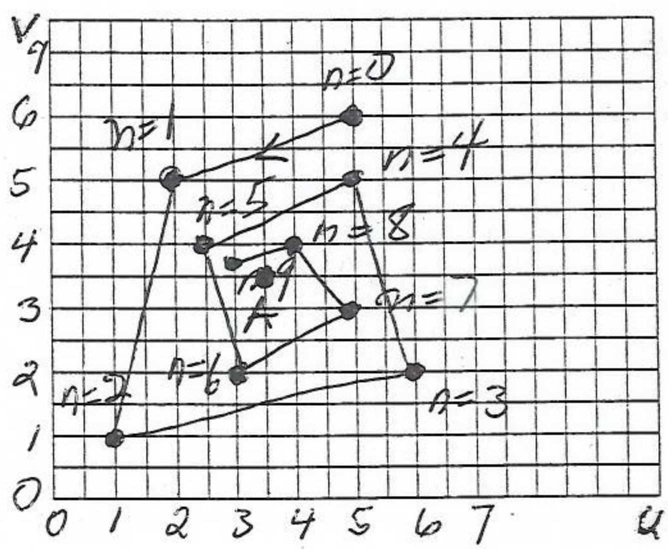
$-7 = -v^*$
$v^* = 7$
$u^* = -4 + 2v^*$
$= 10$

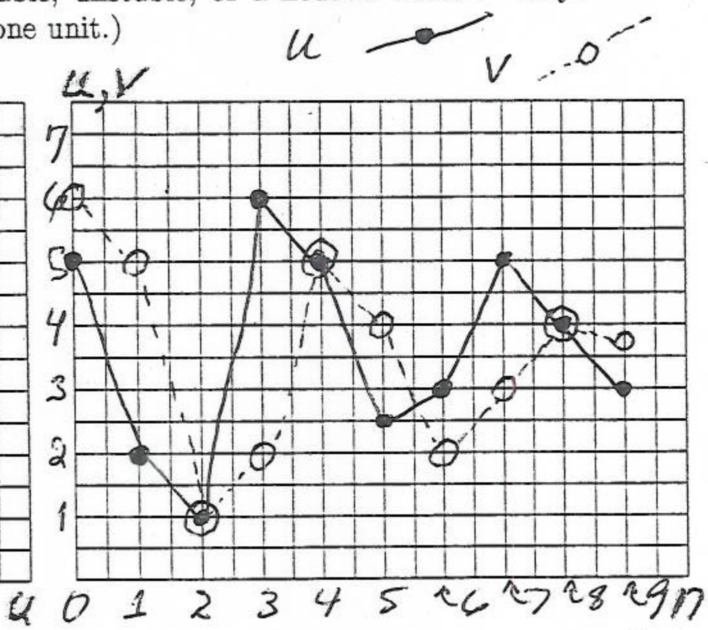
$\boxed{(10, 7)}$

4. (15 pts) Here is a table of values for a 2-variable discrete system. The point A $(3.5, 3.5)$ is an equilibrium.

n	0	1	2	3	4	5	6	7	8	9
u_n	5	2	1	6	5	2.5	3	5	4	3
v_n	6	5	1	2	5	4	2	3	4	3.75

Plot u_n and v_n against one another on one graph, and label the pts with the values of n from 0 to 9. Plot u_n and v_n on a single graph against n from 0 to 9. Is the equilibrium A is stable, unstable, or a neutral center? Why? (Suggestion: let 2 boxes represent one unit.)





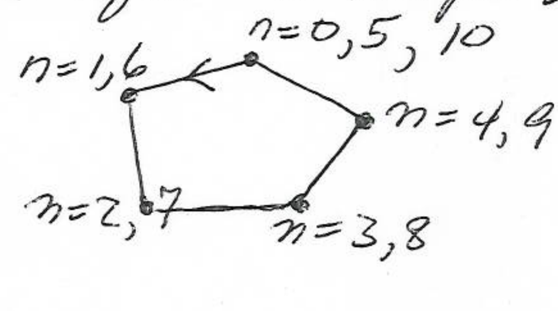
A appears to be a stable equil. because (u_n, v_n) is spiralling in to A. The oscillations of u_n and v_n are damped.

5. (5 pts) A discrete two variable system is said to have a 5-cycle. What does this mean about the (u_n, v_n) coordinates? Illustrate graphically (a sketch suffices; you don't need to give precise numbers).

$(u_0, v_0) = (u_5, v_5) = (u_{10}, v_{10})$ etc

$(u_1, v_1) = (u_6, v_6) = (u_{11}, v_{11})$ etc

Coord's repeat after 5 steps.



6. (12 pts) A matrix M has eigenvectors $\mathbf{e}_1 = \begin{bmatrix} 11 \\ 5 \\ 4 \end{bmatrix}$, $\mathbf{e}_2$, and $\mathbf{e}_3$. These go with eigenvalues $\lambda_1 = 1.05$, $\lambda_2 = -0.9$, and $\lambda_3 = 0.1$, respectively. Let $\mathbf{u}_0 = 3\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$. The matrix M is a population projection (Leslie-Lefkowitch) matrix.

a. Compute $M^2\mathbf{u}_0$. You may leave the symbols $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in your answer, but the rest should be numerical.

$$M^2\vec{u}_0 = MM\vec{u}_0 = M\vec{u}_1$$

$$\vec{u}_1 = M\vec{u}_0 = 3M\vec{e}_1 - M\vec{e}_2 + M\vec{e}_3$$

$$= 3(1.05)\vec{e}_1 - (-0.9)\vec{e}_2 + 0.1\vec{e}_3$$

$$\vec{u}_2 = M\vec{u}_1 = 3(1.05)^2\vec{e}_1 - (-0.9)^2\vec{e}_2 + (0.1)^2\vec{e}_3$$

b. Rewrite $M^n\mathbf{u}_0$ in the form $a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3$, where a, b, and c involve numbers and n. What happens to this quantity as n gets larger and larger?

$$\vec{u}_n = M^n\vec{u}_0 = 3(1.05)^n\vec{e}_1 - \underbrace{(-0.9)^n}_{0}\vec{e}_2 + \underbrace{(0.1)^n}_{0}\vec{e}_3$$

as $n \to \infty$

$$\approx 3(1.05)^n\vec{e}_1$$

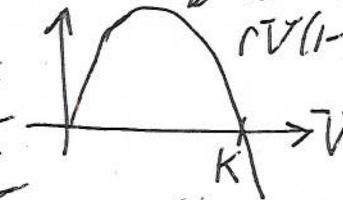
longterm $\vec{u}_n$ is a multiple of $\vec{e}_1$, so we have SAD.

c. Describe the long term rate of growth/decline of the population. What is the stable age distribution of the population, or is there no such thing? Briefly explain.

Eventually the population grows just by multiplying by the dominant e-val, $\lambda_1 = 1.05$, This represents a 5% growth rate. The dominant e-vect $\vec{e}_1 = \begin{bmatrix} 11 \\ 5 \\ 4 \end{bmatrix}$ gives the SAD $\begin{bmatrix} 11/20 \\ 5/20 \\ 4/20 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.25 \\ 0.2 \end{bmatrix}$.

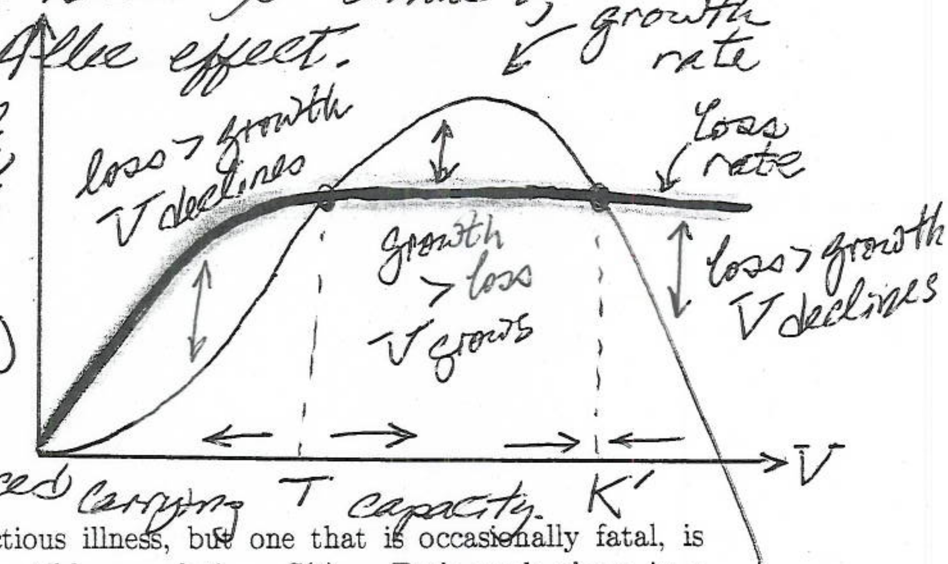
7. (12 pts) A victim population has a growth rate curve (light line) as shown.
   a. At low population levels the growth rate is not logistic; how is it different?
   b. Superimposed on this graph is a heavy curve indicating the loss rate due to moderate presation by a predator that exhibits a type II functional response. Label the equilibrium values for V^* on the graph, determine if each is stable or unstable, and indicate verbally or by arrows how V will change if it falls just slightly off each equilibrium value.

$\frac{dV}{dt} = rV(1-\frac{V}{K})$



(a) Logistic has a parabola shape: growth starts out fast for small V. Here growth is slow for small V, also called the Allee effect.

(b) T is unstable, called a threshold equilibrium (or critical min. pop.) K' is stable and functions as a reduced carrying capacity.



8. (12 pts) A not very infectious illness, but one that is occasionally fatal, is spreading through a susceptible population $S(t)$. Each week there is a mass action interaction with transmission coefficient 0.04 of the susceptible population $S(t)$ with the ill and infectious population $I(t)$ in which susceptible individuals become ill and infectious. At the same time 33% of the infectious population recovers; the recovered population is denoted $R(t)$. Just 0.5% of the infected population dies. Most of the recovered population, in fact 92%, is immune to reinfection, but the rest do become susceptible again. Write a system of **continuous** model equations for for this process (the unit of measurement is thousands of people per week).

loss to S is gain to I

$$\frac{dS}{dt} = -0.04SI + 0.08R$$

$$\frac{dI}{dt} = +0.04SI - 0.005I - 0.33I$$

loss to R is gain to S

$$\frac{dR}{dt} = 0.33I - 0.08R$$

gain to R is loss to I

9. (12 pts) Consider the following continuous model of a predator-prey system.

$$\frac{dV}{dt} = 0.5V(1-\frac{V}{250}) - 0.02VP = V\left(0.5(1-\frac{V}{250}) - 0.02P\right)$$
$$\frac{dP}{dt} = -0.8P + 0.004VP = P(-0.8 + 0.004V)$$

a. What kind of growth does the victim population exhibit if there are no predators (i.e., $P = 0$)? Why is $(V^*, P^*) = (250, 0)$ an equilibrium, and how do you interpret this biologically?

If $P=0$, $\frac{dV}{dt} = 0.5V(1-\frac{V}{250})$ Logistic growth $r = 0.5$ $K = 250$

Notice if $V^* = 250$ and $P^* = 0$, then $\frac{dV}{dt} = 0$, $\frac{dP}{dt} = 0$. So this is an equil with V at carrying capacity and no predators.

b. Compute the equilibrium (V^*, P^*) other than $(0,0)$ and $(250,0)$ for this system.

$0 = -0.8 + 0.004V^*$ (from $\frac{dP}{dt} = 0$)

gives $\boxed{V^* = 200}$.

$0 = 0.5(1-\frac{V^*}{250}) - 0.02P^* = (0.5)(1-\frac{200}{250}) - 0.02P^*$

from $\frac{dV}{dt} = 0$

$= (0.5)(1-\frac{4}{5}) - 0.02P^*$

$= 0.1 - 0.02P^*$ gives $\boxed{P^* = 5}$

c. (Bonus) Mark the coordinates of the equilibrium pts (heavy ~~dots), and~~ place the predator population arrows (up or down), victim population arrows (left or right), and net population change arrows at the open dots.

Left of the vertical line $\frac{dP}{dt} < 0$ and so P decreases. The opposite to the right. Below the slanted line, $\frac{dV}{dt} > 0$ so V increases; opposite above.