MATH 172 Fall, 2009 Exam #2 Name: M^2

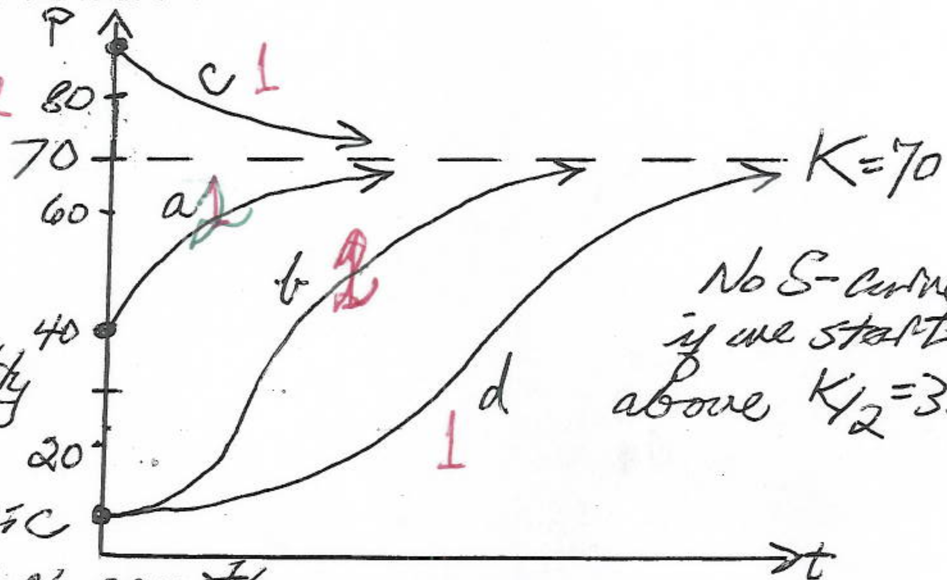
For full credit you must show sufficient work to justify your answer. You may use the following: (1) an affine continuous model $\frac{dQ}{dt} = aQ + b$ has an explicit solution $Q(t) = Ce^{at} + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition; (2) an affine discrete model $Q_{n+1} = aQ_n + b$ has an explicit solution $Q_n = Ca^n + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition.

#3 & #4 have been exchanged, & typos corrected.

1. (11 points) Sketch all the graphs of $P(t)$ if $\frac{dP}{dt} = rP(1 - \frac{P}{70})$. Label your graphs **clearly** with a, b, c, d.
   a. if $r = 0.4$ and $P(0) = 40$
   b. if $r = 0.4$ and $P(0) = 10$
   c. if $r = 0.2$ and $P(0) = 90$
   d. if $r = 0.2$ and $P(0) = 10$
   e. What is this kind of model called? What is the biological significance of the number 70? of the value of r?

start the same, but (b) has faster growth than (d)

$\frac{dP}{dt} = 0$ for $P^* = 0$, $P^* = 70$. 2



1 This is a logistic model; 70 is 1 the carrying capacity and r is the 1 natural or intrinsic or per capita rate of growth.

No S-curve if we start above $K/2 = 35$.

2. (11 points) Let $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Which of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is an eigenvector, and which is not? Explain, and give the corresponding eigenvalues, where appropriate.

$A\vec{u} = \begin{bmatrix} -20 \\ 24 \end{bmatrix}_1 = -4\begin{bmatrix} 5 \\ -6 \end{bmatrix} = -4\vec{u}$ 1 $\vec{u}$ is eigenvector with eigenvalue $\lambda = -4$ 2

$A\vec{v} = \begin{bmatrix} 13 \\ -3 \end{bmatrix}$ is not a multiple of $\vec{v}$, so $\vec{v}$ is 2 not an eigenvector

$A\vec{w} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}_1 = 7\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 7\vec{w}$ 1 $\vec{w}$ is an eigenvector with eigenvalue $\lambda = 7$ 2

3. (11 points) A fish population $F = F(t)$, measured in thousands, is governed by the continuous model $\frac{dF}{dt} = 0.5F(1 - \frac{F}{100})(\frac{F}{25} - 1) = 0.0002F(100 - F)(F - 25)$ in thousands of fish per year. Sketch the graphs of $F(t)$, and label them **clearly** if (a) $F(0) = 20$, (b) $F(0) = 120$, (c) $F(0) = 30$.

d. Determine all the equilibrium values for F, and state whether each one is stable or unstable.

e. What is this kind of model called? What is the biological significance of the each equilibrium value?

stable, but biologically uninteresting

$\frac{dF}{dt} = 0$ at $F^* = 0$, 1

$F^* = 100$, 1 $F^* = 25$ 1

This is a logistic

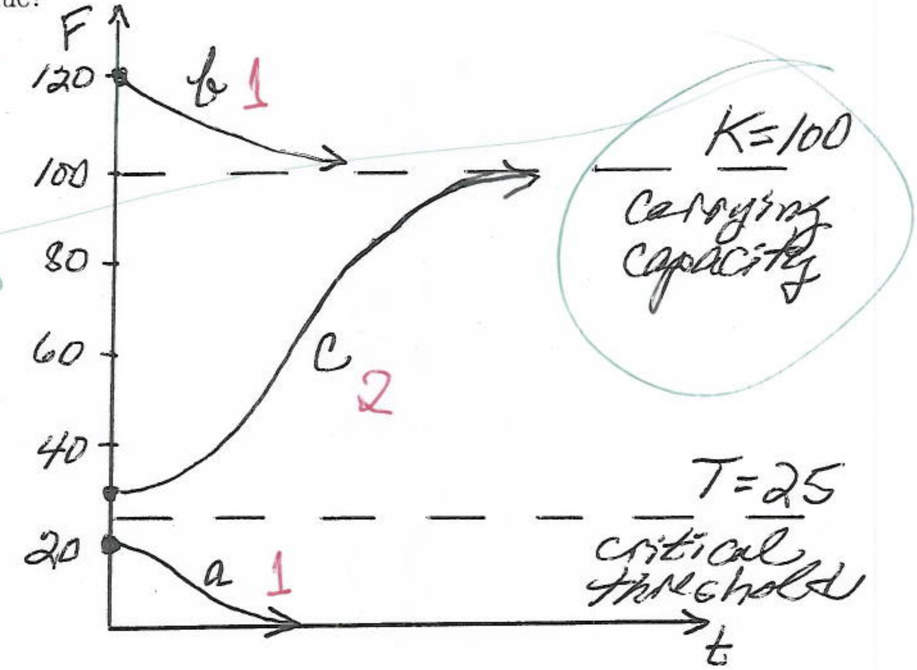
1 model ($F^* = K = 100$

is a stable 1 equil.)

with a critical

1 threshold ($F^* = T = 25$ is unstable)

1



4. (15 points) A honey bee population $H = H(t)$, measured in millions, now finds that the quality of the habitat is decreasing as measured by the per capita growth rate $r(t)$; in fact $\frac{dH}{dt} = r(t)H$, with $r(t) = 0.072 - 0.006t$ in units of yr^{-1}. At time $t = 0$ there were 30 million individuals. ← so we may as well assume $H > 0$.

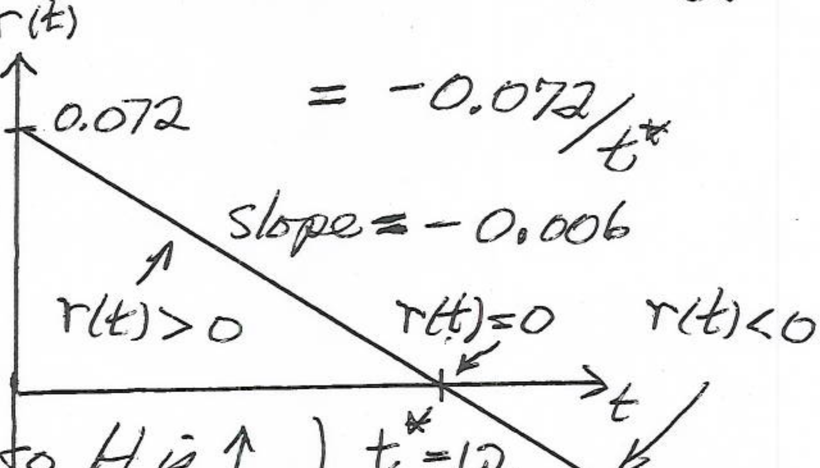
a. How does the information provided tell us that the quality of the habitat is decreasing? At what time does the habitat switch from being a positive growth environment to a negative growth environment; explain how you can recognize these features.

5

2 Because the per capita growth rate is declining.

$-0.072 = -0.006\,t^*$

2 $t^* = 12$ years



$\frac{dH}{dt} > 0$ for $t < 12$, so H is ↑

$\frac{dH}{dt} < 0$ for $t > 12$, so H is ↓

$t^* = 12$

Since $\frac{dH}{dt}$ has the same sign as $r(t)$.

1

So the oscillations get smaller & smaller 2
and $Q_n \to 50$ as $n \to \infty$ (stable equil) 2

b. Solve the model equation and use your solution to find the number of bees at $t = 20$ years. Bonus: at what time is the number of bees at a maximum, and how many are there?

10

Separate the variables.

$$\frac{dH}{dt} = r(t)H = (0.072 - 0.006t)H$$

$$\int \frac{dH}{H} = \int (0.072 - 0.006t)\,dt \quad 3$$

$$\ln H = 0.072t - 0.006\frac{t^2}{2} + C \quad 4$$

$$H(t) = Ae^{0.072t - 0.003t^2} \quad 1$$

A is some const, that we get from:

$$30 = H(10) = Ae^0 = A \quad 1$$

$$\boxed{H(t) = 30e^{0.072t - 0.003t^2}} \quad 1$$

$\boxed{H(20) = 38.1 \text{ mill}}$

We have seen that $H(t)$ is increasing for $t<12$, $\frac{dH}{dt} = 0$ at $t=12$, and $H(t)$ is decreasing for $t>12$. So the max is $H(12) = 46.2$ million bees. +3

5. (16 points) We are given a discrete model $Q_{n+1} = (-0.9)Q_n + 95$ with $Q_0 = 70$.

a. Find the explicit solution for Q_n.

b. What happens to Q_n as $n \to \infty$? Does it increase, decrease, oscillate, tend towards or away from the equilibrium? Conclude whether the equilibrium is stable or not.

First find the equil Q^*: $Q^* = Q_{n+1} = Q_n$

$$Q^* = -0.9Q^* + 95$$

$$1.9Q^* = 95$$

$$Q^* = 50 \quad \cancel{3}\ 4$$

Solution has form $Q_n = C(-0.9)^n + 50$. $\cancel{4}$ 3

$70 = Q_0 = C(-0.9)^0 + 50$, so $C = 20$.

$$\boxed{Q_n = 20(-0.9)^n + 50} \quad 2$$

$Q_0 = 70$, $Q_1 = 32$, $Q_2 = 66.2$, $Q_3 = 35.4$, 2

and in general Q_n oscillates above and below $Q^* = 50$. But $(-0.9)^n \to 0$ as $n \to \infty$

6. (16 points) In a good, but shrinking, habitat, a population of snails $S = S(t)$ growing at a **per capita rate** of 2% yr $^{-1}$, but 50 migrate away over the course of each year. Write an affine continuous model equation for this situation, and solve it, assuming that the initial snail population is 2,000. What exactly happens to the snail population in the long term, and how do you know? If the population is growing, compute the doubling time; if the population is shrinking, compute the extinction time.

$$\frac{dS}{dt} = 0.02S - 50 \quad 5$$

Equil: $0 = 0.02S^* - 50$

$S^* = 2500 \quad 3$

Solution has form $S(t) = Ce^{0.02t} + 2500$.

$2000 = S(0) = Ce^{0} + 2500$, so $C = -500$, 2

$$\boxed{S(t) = 2500 - 500e^{0.02t}} \quad 2$$

growth exponential

2 Since more and more is subtracted from 2500 as t grows, there is eventual extinction: $S(t) = 0$ when

$2500 = 500e^{0.02t}$, $5 = e^{0.02t}$, $t = \frac{\ln(5)}{0.02} = 80.5$ years

2

7. (20 points) A population consists of individuals in four stages of development: newborns (N_t), juveniles (J_t), reproductive adults (R_t), and post-reproductive adults ("grandmothers" G_t). Newborns have a mortality rate of 70%; those that survive become juveniles. Juveniles have a total survival rate of 60% in two categories: 20% remain in the juvenile phase and 40% advance to the reproductive adult phase. Also juveniles rarely reproduce: on average each contributes 1 newborn to the next generation. Reproductive adults have a survival rate of 80% as reproductive adults, and 10% become post-reproductive adults. Meanwhile each contributes 5 newborns to the next generation. Post-reproductive adults produce no offspring, but have a survival rate of 70%.

a. Set up the population transition matrix A to express the information above, so that $\mathbf{P}_{t+1} = A\mathbf{P}_t$.

2

$$A = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0.3 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0.8 & 0 \\ 0 & 0 & 0.1 & 0.7 \end{bmatrix}$$

1 1 1

b. The initial population vector is $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 10 \\ 10 \\ 10 \end{bmatrix}$. Compute $\mathbf{P}_1$.

3

$$\vec{P}_1 = \begin{bmatrix} 10+50 \\ 30+2 \\ 4+8 \\ 1+7 \end{bmatrix} = \begin{bmatrix} 60 \\ 32 \\ 12 \\ 8 \end{bmatrix}$$

c. At $t = 10$ we have $\mathbf{P}_{10} = \begin{bmatrix} 883 \\ 238 \\ 185 \\ 30 \end{bmatrix}$. Determine the total population and the distribution vector $\mathbf{D}_{10}$. Carry three decimal point accuracy. 1

Total = 1336

2

$$\vec{D}_{10} = \begin{bmatrix} 0.661 \\ 0.178 \\ 0.138 \\ 0.0225 \end{bmatrix}$$

Alternatively compute the top entry of $\vec{P}_{11} = A\vec{P}_{10}$. This way we get

$$N_{11} = (1)(238) + (5)(185) = 1163.$$

d. The dominant eigenvalue is $\lambda = 1.3147$ with eigenvector $\mathbf{v} = \begin{bmatrix} 2906 \\ 782 \\ 608 \\ 99 \end{bmatrix}$.

Has the population reached its stable age/stage distribution at $t = 10$? How can you tell? Predict the total population at $t = 11$ and the value of N_{11}.

Total for $\vec{v}$ is 4395. $SAD = \frac{1}{4395}\vec{v} = \begin{bmatrix} 0.661 \\ 0.178 \\ 0.138 \\ 0.0225 \end{bmatrix}$

2 Since this agrees with $\vec{D}_{10}$ we know $\vec{P}_{10}$ is at SAD.

4

Then everything grows by the same factor $\lambda = 1.3147$.

1

At $t=11$, total pop $= (1.3147)(1336) = 1756$;

$N_{10} = 883$ so $N_{11} = (1.3147)(883) = 1161$, approx. 2

9. (Bonus: 10 points) With a certain drug, we find that 60% of drug in the bloodstream is used up from one day to the next, so the patient takes a daily maintenance dose of 30 mg.

   a Formulate an affine discrete model in which A_n denotes the amount of drug present in the bloodstream on day n.

   b. Give an explicit formula for A_n if the initial dose is $A_0 = 60$ mg.

   c. Compute A_1, A_2, A_5, A_{20} either from the formula of part (b) or from the model of part (a) with the help of your calculator.

   d. Describe the long term behavior of A_n; does it increase, decrease, oscillate, tend towards or away from the equilibrium? Does the equilibrium value you found in part (b) appear to be stable or unstable? Explain verbally and / or graphically.

(a) $A_{n+1} = A_n - 0.6A_n + 30$

4 $= 0.4A_n + 30$ (40% is still in bloodstream)

(b) Find $A^* = A_{n+1} = A_n$. $A^* = 0.4A^* + 30$

1 $0.6A^* = 30$ $A^* = 50$

$A_n = C(0.4)^n + 50$

$60 = A_0 = C(0.4)^0 + 50 = C + 50$

$+10 = C$

1 $\boxed{A_n = 10(0.4)^n + 50}$

(c) $A_0 = 60$, $A_1 = 54$, $A_2 = 51.6$,

2 $A_5 = 50.1$, $A_{20} = 50$

(d) A_n decreases to the equil value of 50 since $(0.4)^n \to 0$ as $n \to \infty$.

2 The equil. appears to be stable.