MATH 172 Fall, 2009 Exam #1 Name: M^2

There are 120 points, which will be scaled down to a percentage (out of 100). **For full credit you must show your work.** All answers should be **correctly rounded to 3 decimal places**, or fewer if that is what is provided in the data.

1. (10 points) If $Q(n) = 5n - n^2$, compute ΔQ.

$$\Delta Q = Q(n+1) - Q(n) = 5(n+1) - (n+1)^2 - (5n - n^2)$$
$$= 5n + 5 - n^2 - 2n - 1 - 5n + n^2$$
$$= -2n + 4$$

2. (10 points) Papua New Guinea currently has a population of 56.7 hundred thousand people. The **net** growth rate is 1.253 hundred thousand people per year. Compute the intrinsic or **per capita** growth rate r. Write a continuous model equation for this process. Give the explicit solution and use your result to predict the population a decade from now.

$$\frac{dP}{dt} = rP$$
$$1.253 = r(56.7)$$
$$\boxed{r = 0.0221}$$

$$\frac{dP}{dt} = 0.022P$$
$$P(t) = P(0)e^{0.0221t}$$
$$P(10) = 56.7e^{(0.0221)(10)}$$
$$\boxed{P(10) = 70.723 \text{ hundred thousand people}}$$

3. (15 points) A population $B(t)$ of bacteria is growing **continuously** over time so that the **per capita** rate of increase is 6% /day.

a. Write the continuous model equation that describes this situation.

$$\frac{dB}{dt} = 0.06B$$

b. If $B(0) = 5$ g, write the solution equation for this model equation. (We don't count bacteria, we weigh them.) This initial population of 5 g grows to what size in 3 weeks?

$$B(t) = B(0)e^{0.06t} = 5e^{0.06t}$$

3 weeks = 21 days

$$\boxed{B(21) = 5e^{(0.06)(21)} = 17.627 \text{ g}}$$

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4. (20 points) A population R of rats kept in a lab depends on t. The experimenter is testing the effect of a poison in the food. Observation takes place at regular weekly intervals. At the start of the experiment ($t = 0$), the population is 400. Week by week, the population shrinks by 9%. ← so $r = -0.09$ is neg.

   a. Write the difference equation for this discrete model.

$$\Delta R = R_{t+1} - R_t = -0.09 R_t$$

   b. Write the updating equation for this model.

$$R_{t+1} = R_t - 0.09 R_t = 0.91 R_t$$

   c. Write the explicit solution for this model, and use this to compute the population after 6 months (26 weeks).

$$R_t = (0.91)^t R_0 = (0.91)^t (400)$$

$$R_{26} = 34.446 \text{ (or 34 rats)}$$

   d. At what time is the population 1/2 of the original?

$$200 = (0.91)^t (400) \quad \text{or} \quad \tfrac{1}{2} R_0 = (0.91)^t R_0$$

$$\tfrac{1}{2} = (0.91)^t$$

$$\ln(\tfrac{1}{2}) = \ln((0.91)^t) = t \ln(0.91)$$

$$t = \frac{\ln(1/2)}{\ln(0.91)} = 7.350 \text{ weeks}$$

5. (10 points) Due to a parasite infecction that increases mortality and decreases reproduction, a host population is decreasing at a continuous rate of 4% yr^{-1}. ~~Infected~~ Uninfected organisms move into the habitat at a rate 100/year and get infected like all the rest. These become new hosts. Write the continuous model equation to describe this process, and compute the equilibrium value, if there is one.

Let $H(t)$ be the host population. $r = -0.04$ neg. because decrease

$$\frac{dH}{dt} = -0.04 H + 100$$

Extra: Equil when $\frac{dH}{dt} = 0$. $0 = -0.04H + 100$

$H = 2500$

This equil appears to be stable. Note that if $H > 2500$, then $\frac{dH}{dt} < 0$ so $H(t)$ decreases; if $H < 2500$, then $\frac{dH}{dt} > 0$ and $H(t)$ increases.

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6. (25 points) A drug is given in pill form (160 mg) once a day. We know that 80% of drug in the bloodstream is used up from one day to the next, but the remainder is reinforced by a maintenance dose of one pill a day. let A_t denote the amount of the drug in mg after t days. The initial dose, given late in the day, is one half pill (80 mg). $\leftarrow \quad A_0 = 80$

a. Formulate a discrete model to describe this process over time. Compute A_1 and A_2.

$$A_{t+1} = A_t - 0.8A_t + 160$$
$$= 0.2A_t + 160 \quad \text{(20\% remains, + dose)}$$
$$A_1 = (0.2)A_0 + 160 = 176 \text{ mg}$$
$$A_2 = (0.2)A_1 + 160 = 195.2$$

b. What is the steady state (equilibrium) amount of the drug in the bloodstream? Look for $A_{t+1} = A_t$ or loss = gain.

$$A_t = 0.2A_t + 160 \text{ or } 0.8A_t = 160 \text{ or } \boxed{A_t = 200.}$$

c. In the calculator version of this model A_t is replaced by $u(n)$. It has nMin = <u>0</u>, $u(n\text{Min}) =$ <u>80</u>, and $u(n) =$ (<u>0.2</u>)u(<u>n−1</u>)+(<u>160</u>).

d. Compute $A_3 = u_3$, $A_5 = u_5$, $A_{10} = u_{10}$ by hand or by calculator or both. What is the long term trend?

$$A_3 = 199.04$$
$$A_5 = 199.96$$
$$A_{10} = 200$$

level of drug approaches steady-state of 200 mg.

d. (Bonus) Is the equilibrium value you found in part (b) stable or unstable? Explain verbally and / or graphically.

The equil. appears to be stable because if we start with $A_0 < 200$, then $A_t \to 200$ as t gets larger.* On the other side if we start with, say, $A_0 = 240$** (or any $A_0 > 200$), then $A_t \to 200$ coming down from above.

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* as seen above ** done on calculator

7. (20 points) An algal population $A(t)$, measured in tons, grows in some lakes and rivers that accumulate nutrients from agricultural runoff. It is governed by the continuous model $\frac{dA}{dt} = 0.0003A(200 - A) = 0.06A(1 - \frac{A}{200})$ tons/year. The initial population is $A(0) = 50$ tons. Since we are measuring in tons, it is reasonable that you may get decimal answers; round off to three decimal places.

   a. Apply stepwise estimation using Euler's method to estimate the population after 12 years. Remember that $\Delta A \approx \frac{dA}{dt}(\Delta t)$. Use 3 steps, so each time step is $\Delta t =$ __4__ to approximate $A(12)$, and show your work! Here is the outline of a table to help you get started.

n	t	$A(t)$	$A'(t)$	$\Delta A \approx A'(t)\Delta t$	$A(t+\Delta t)$
0	0	50	2.25	9	59
1	4	59	2.496	9.983	68.983
2	8	68.983	2.711	10.856	79.829
3	12	79.829			

$$\boxed{\text{So } A(12) \approx 79.829 \text{ tons}}$$

Extra: If you look at $A'(t)$ or $\frac{\Delta A}{\Delta t}$, you'll see that it is positive as long as $A < 200$, so $A(t)$ will continue to increase.

   b. If we want to approximate $A(12)$ again, but use $\Delta t = 1/4 = 0.25$ year (that is, 3 months), how many steps are required to get to 12 years?

48.

$N = \#$ steps

$$0.25 = \Delta t = \frac{12}{N}$$

$$\boxed{N = \frac{12}{0.25} = 48}$$

Exam stats: 31 exams, mean = 61, median = 59, SD = ±25, low = 10, high = 107.