Derivation and Comparison of Radiation Boundary Conditions for the Two-Dimensional Helmholtz Equation with Non-Circular Artificial Boundaries*

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Abstract

Wave equations in exterior domains typically include a boundary condition at infinity to ensure the well-posedness of the problem. An obstacle to the efficient computation of solutions is the unbounded computational domain; the problem must be reformulated on a bounded domain. The success of this approach depends critically on the selection of the artificial boundary and on the radiation boundary condition imposed on the artificial boundary. This choice involves a compromise between accuracy of the reformulation and efficiency of solution.

Several radiation boundary conditions have been proposed; most are designed to be used on simple boundaries, primarily circles. In scattering problems with a long, thin scatterer a circular artificial boundary results in a relatively large computational domain. Smaller computational domains can be obtained by selecting an artificial boundary which is more conformal with the scatterer.

We demonstrate a general method for extending radiation boundary conditions to a smooth and convex boundary. Additional approximations are made to balance accuracy of the solution with the computational effort needed to find the solution. The results from this method are compared with other boundary conditions previously proposed.

1 Introduction

The solution of the wave equation in an exterior domain is a common problem in electromagnetics and other disciplines in the scattering of waves is studied. The unbounded domain complicates the search for a numerical approximation to the solution. A number of different approaches to this problem have been proposed, studied and implemented. A common feature to many of these procedures is the approximate reformulation of the problem on a bounded domain. The solution of this problem is then possible by standard numerical methods. The papers by Givoli [6] and Moore *et al.* [14] provide good summaries of much of the work that has been done on this problem.

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Engquist and Majda ([4], [5]) conducted some of the earliest work on artificial boundary conditions. Their techniques, which are based on the theory of pseudo-differential operators, are most effective for a normally incident wave impinging on a planar artificial boundary. These ideas have been extended and generalized in numerous ways, in particular by Higdon [7]. Radiation boundary conditions designed for use on a circular artificial boundary have been introduced by Bayliss and Turkel ([1], [2]). These boundary conditions are based on the annihilation of leading terms in an asymptotic expansion of the solution in the far field. Similar ideas making use of a modal expansion have been recently proposed by Li and Cendes [11]. The Bayliss-Turkel and Li-Cendes RBCs have the same general form. Time-dependent problems have been considered by several groups, including Engquist and Halpern [3].

For this discussion we concentrate on the Bayliss-Turkel and Li-Cendes boundary conditions in two spatial dimensions. The restriction that the artificial boundary be a circle is overly restrictive. For example, a circumscribing circle for a long and thin scatterer results in a large computational domain. Because the artificial boundary conditions are based on asymptotic expansions, the portions of the artificial boundary closest to the scatterer typically dominate the approximation error. This suggests the use of artificial boundaries located at a more uniform distance from the scatterer. To implement this idea it is necessary to obtain radiation boundary conditions for a general curved artificial boundary. The extension of radiation boundary conditions from a circle to a general smooth artificial boundary is the main topic of this discussion.

The remainder of this paper is organized as follows. Brief discussions of the problem on the unbounded domain and of the radiation boundary conditions for a circular artificial boundary are presented in Sections 2 and 3, respectively. Section 4 contains the details behind the extension to a general artificial boundary. We conclude, in Section 5, with a comparison between the new RBCs and those previously proposed by other researchers.

2 The Problem on an Unbounded Domain

The problem of interest is to find the scattered wave produced when an incident wave is reflected from a scatterer. The incident field is assumed to be time-harmonic with a temporal dependence of $e^{-i\omega t}$, where ω denotes the frequency of the incident field. Assuming the wave speed is c, the dimensionless wave number is $k = \frac{\omega}{c}$. Thus, the mathematical problem is to find the solution to

$$\begin{aligned}
&(1) & \triangle u + k^2 u &= 0 && \text{in } \Omega^+ \\
&(2) & u &= g && \text{on } , \\
&(3) & \lim_{r\to\infty} r^{1/2}(u_r - iku) &= 0 && \text{uniformly in } \theta
\end{aligned}$$

where , is the boundary of the scatterer, Ω^+ is the (unbounded) exterior of the scatterer, and g is obtained from the incident wave.

The "boundary condition at infinity", (3), is the Sommerfeld radiation condition in two spatial dimensions. The purpose of this condition is to guarantee that the problem has a unique solution and, more generally, that the problem is well-posed. Successful reformulation of the exterior problem (1), (2), (3) into a problem on a bounded domain hinges on the selection of an appropriate artificial boundary, , ^{t}, and boundary operator

$\mathcal{B}$ to replace the Sommerfeld radiation condition. In particular, this boundary condition must have the property that waves arriving at $,^t$ from inside the computational domain are passed through the artificial boundary without generating spurious reflections back into the truncated domain, Ω^t. The reformulated problem thus appears as

$$\begin{aligned} \triangle u + k^2 u &= 0 \quad \text{in } \Omega^t && (4) \\ u &= g \quad \text{on } , && (5) \\ \mathcal{B}u &= 0 \quad \text{on } ,^t && (6) \end{aligned}$$

where $\mathcal{B}$ is the, as yet unspecified, radiation boundary operator.

An exact boundary condition with this property does exist:

$$\widetilde{\mathcal{B}}u = \frac{\partial u}{\partial n} - Mu,$$

where M is the Dirichlet-to-Neumann map [9]. This boundary condition is easily implemented in a finite element method. However, because the boundary operator M is non-local in space, the resulting system of linear equations is not sparse. The extra computational effort needed to assemble and solve the system of equations detracts from the use of this boundary condition.

To avoid these difficulties we restrict our attention to spatially local boundary conditions. Because of the general domains on which we are working, we also require the boundary condition to be in a form that is appropriate for the finite element method.

3 RBCs for a Circular Artificial Boundary

A common form for approximate RBCs on a circle is

$$\mathcal{B}u = u_r - \alpha(r)u - \beta(r)u_{\theta\theta}, \tag{7}$$

where (r, θ) are the polar coordinates and α and β are appropriately selected radial functions. Note that (7) is a local boundary condition. Moreover, implementation of this boundary condition in a variational (FEM) method is very natural.

Selection of the coefficients α and β is often based on an expansion of the solution in the exterior of a circle. A sequence of approximate RBCs, $\{\mathcal{B}_m\}$, can be constructed from the boundary operators that annihilate an increasing number of the leading terms in the expansion. Two examples are the Bayliss-Turkel RBCs [2], based on the far-field expansion

$$u = \frac{e^{-ikr}}{\sqrt{r}} \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n} \tag{8}$$

and the Li-Cendes RBCs [11], based on the Hankel expansion

$$u = \sum_{n=0}^{\infty} b_n(\theta) H_n(kr),$$

where $H_n := H_n^{(2)}$ is the n^{th}-order Hankel function of the second kind. A third set of coefficients, the Mittra-Ramahi RBCs [10] are simply the $O(r^3)$ expansions of the second-order Bayliss-Turkel RBCs. The coefficients for the first- and second-order RBCs for all three of these methods are listed in Table 1.

TABLE 1. *Coefficients of first- and second-order RBCs for a circular boundary. The functions g_1 and g_2 which appear in the coefficients of the second-order Li-Cendes RBC are rational functions of H_0, H_1 and their first two derivatives, see [4] for explicit formulae.*

Type	α_1	β_1	α_2	β_2
Bayliss-Turkel	$-ik - \frac{1}{2r}$	0	$-ik - \frac{1}{2r} + \frac{1}{8r(1+ikr)}$	$\frac{1}{2r(1+ikr)}$
Li-Cendes	$-\frac{kH_1(kr)}{H_0(kr)}$	0	$-\frac{k^2r(g_2(kr)-1)}{(krg_1(kr)-1)}$	$\frac{1}{(krg_1(kr)-1)}$
Mittra-Ramahi	$-ik - \frac{1}{2r}$	0	$-ik - \frac{1}{2r} - \frac{i}{8kr^2} + \frac{1}{8k^2r^3}$	$-\frac{i}{2kr^2} + \frac{1}{2k^2r^3}$

The terminology used above is standard, but somewhat confusing. The order of the RBC is related to both the number of terms annihilated in the appropriate expansion and the asymptotic accuracy of the approximate solutions. For example, the order m Bayliss-Turkel RBC on a circular artificial boundary with radius R annihilates $2m$ terms in (8). The associated error between the exact solution to the problem on an unbounded domain and the solution to the problem on the truncated domain is $\mathcal{O}(R^{-(2m+1/2)})$ as $R \to \infty$.

4 RBCs on a General Artificial Boundary

When the artificial boundary is not a circle, the radial and angular derivatives in (7) are not convenient for computation. Several translations of (7) into an RBC expressed in terms of normal (ν) and tangential (τ) derivatives on the artificial boundary have been proposed ([8], [10], [13]). The first two introduce approximations that simplify their numerical implementation. In this paper we demonstrate that the exact translation can be used to obtain more accurate RBCs for a general artificial boundary.

The coordinate transformation between polar coordinates (r, θ) and curvilinear coordinates (ν, τ) extends (7) to a general, smooth, artificial boundary. This leads to a boundary condition involving u and all its first- and second-order partial derivatives. Two terms, $u_{\nu\nu}$ and $u_{\tau\nu}$, cause particular concern when viewed with the intent of implementation in a finite element method. The second normal derivative, $u_{\nu\nu}$, can be re-expressed in terms of "good" terms via the curvilinear form of the Helmholtz equation. At this point the approximate radiation boundary condition is

$$\mathcal{B}u = Au + Bu_\tau + Cu_\nu + Du_{\tau\tau} + Eu_{\tau\nu} \tag{9}$$

where

$$\begin{aligned} A &= \alpha - k^2\beta(\nu_\theta)^2, \\ B &= -\tau_r + \beta\tau_{\theta\theta}, \\ C &= -\nu_r + \beta(\nu_{\theta\theta} - \kappa(\nu_\theta)^2), \\ D &= \beta((\tau_\theta)^2 - (\nu_\theta)^2), \\ E &= 2\beta\tau_\theta\nu_\theta. \end{aligned} \tag{10}$$

In these formulae the coordinate transformation is represented by $\nu = \nu(r, \theta)$, $\tau = \tau(r, \theta)$ and $\kappa = \kappa(\tau)$ denotes the curvature.

The mixed derivative term, $u_{\tau\nu}$, is not so simple to handle. An exact representation for $u_{\tau\nu}$ can be obtained from the tangential derivative of the curvilinear Helmholtz equation. However, this introduces third-order derivatives, some of which are equally difficult to implement. There is no exact expression for $u_{\tau\nu}$ in terms of u, u_τ, u_ν, and $u_{\tau\tau}$.

The authors ([12], [13]) have previously considered several approximate expressions for the mixed-derivative. The basic idea is to obtain an approximation to $u_{\tau\nu}$ by taking a tangential derivative of a first-order curvilinear RBC. Write the curvilinear first- and second-order RBCs as

$$\begin{aligned} \mathcal{B}_1 u &= A_1 u + B_1 u_\tau + C_1 u_\nu, \\ \mathcal{B}_2 u &= A_2 u + B_2 u_\tau + C_2 u_\nu + D_2 u_{\tau\tau} + E_2 u_{\tau\nu}, \end{aligned}$$

respectively. Then the boundary condition

$$\mathcal{B}_{1,2} u := \mathcal{B}_2 u - \frac{E_2}{C_1}\frac{\partial}{\partial\tau}(\mathcal{B}_1 u) \tag{11}$$

has the form of (9), but without the mixed-derivative term. While the initial idea was to use the first- and second-order Bayliss-Turkel RBCs, this procedure is much more general. It can be applied to any pair of boundary operators, $\mathcal{B}_1$ and $\mathcal{B}_2$, provided $\beta_1 = 0$.

Khebir, Ramahi, and Mittra [10] also noticed the difficulties caused by the mixed-derivative term. Their solution is to use an approximate change of variables chosen so that RBC is of the desired form. As a result, none of the coefficients in the RBC are correct, and the errors are difficult to trace. The final RBC is of the form (9) with coefficients

$$A = \alpha r_\nu, \qquad B = \tau_\theta \theta_\nu, \qquad C = -1, \qquad D = \beta r_\nu (\tau_\theta)^2, \qquad E = 0 \tag{12}$$

A third idea, described in [8], is to use the substitutions

$$\frac{\partial}{\partial r} \mapsto \frac{\partial}{\partial \nu}, \qquad \frac{1}{r}\frac{\partial}{\partial\theta} \mapsto \frac{\partial}{\partial\tau}, \qquad \frac{1}{r} \mapsto \kappa \tag{13}$$

to transform (7) into an approximate RBC in the form of (9). This simple method avoids any problems associated with the mixed-derivative term. However, a comparison with the coefficients for the exact curvilinear RBC, (10), shows that this also brings new approximations into the RBC.

5 Comparison of Approximate RBCs

The approximations introduced in the conversion of the mixed-derivative term into a usable form prevent any of the above approximate RBCs from being second-order in the same sense as the original RBCs. The authors are in the process of analyzing the precise approximation properties of these boundary conditions. Fortunately, the basic numerical performance of the different boundary conditions can be illustrated with a simple test.

Given an artificial boundary , ^{t} and a function u defined in R^2, the exact radiation boundary operator satisfies $\widetilde{\mathcal{B}}u = 0$ on , ^{t}. The approximate RBCs can be compared by computing $\mathcal{B}u$ on , ^{t} for different operators $\mathcal{B}$.

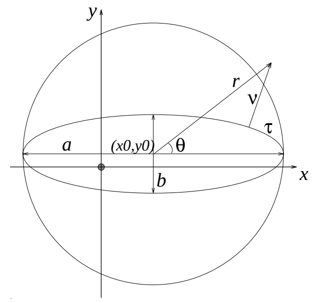

FIGURE 1. *Computational domains for circular and elliptical artificial boundaries.*

Let u be one term from the Hankel expansion, centered at the origin, for the solution to the Helmholtz equation, i.e. $U_\nu(r,\theta) := e^{-i\nu\theta}H_\nu^{(2)}(kr)$, $n = 0, 1, \ldots$. and choose the artificial boundary to be an ellipse with radii a and b and center (x_0, y_0) — see Figure 1. Two different elliptical artificial boundaries are used in the tests. The first ellipse has $a = 5\lambda$, $b = 1\lambda$, and $(x_0, y_0) = (0, 0)$; the second ellipse is larger, but not centered at the origin: $a = 10\lambda$, $b = 2\lambda$, and $(x_0, y_0) = (1\lambda, 1\lambda)$. The results are similar for a wide variety of different artificial boundaries.

Figure 2 displays the errors generated by $u = U_0$ for five different boundary operators using the Bayliss-Turkel coefficients. The two solid curves are the errors for the exact first- and second-order Bayliss-Turkel operators (since u is defined on R^2, computing $u_{\tau\nu}$ is not a problem). The remaining three curves correspond to the boundary operators with coefficients based on (11), (12), and (13). Only the first of these (marked with $\times$) comes reasonably close to the second-order RBC; the other two are more comparable to the first-order RBC. In fact, neither of these last two boundary operators is as good as the RBC obtained when the mixed-derivative is ignored and all remaining coefficients are taken from the exact coordinate transformation, i.e. using (10) with $E = 0$.

The Li-Cendes RBCs can also be used in (11). Since these are perfectly absorbing for the first terms of the Hankel expansion, we use $u = U_2$. Figure 3 compares the exact and approximate RBCs obtained by using the coefficients of the Bayliss-Turkel and Li-Cendes circular RBCs for a circle. The first-order RBCs (which are not shown) are virtually indistinguishable; the pairs of original and approximate second-order RBCs are also quite close. Moreover, the approximate RBCs still do a reasonable job of approximating the original RBCs.

The above comments apply to both test boundaries. The error curves for other artificial boundaries (ellipses as well as other geometries) all convey the same general message: effective radiation boundary conditions for general boundaries can be constructed if appropriate care is used in the coordinate transformation.

6 Conclusion

In this brief report we have presented a general method for extending an RBC from a circular artificial boundary to a general smooth boundary. While the analysis is not complete, the new approach appears to provide more accurate approximations to the classical boundary operators than have been used previously.

The key feature of this method is the elimination of approximations from the extension of the RBC from a circular artificial boundary to a general convex boundary. The benefits of this approach include improved accuracy in the approximation of the exact artificial boundary condition and enough flexibility to permit the design of a customized boundary operator for a specific radiation pattern. The best RBC for a particular problem will depend on the geometry of the scatterer and other parameters in the problem.

The approximate RBC (11) obtained with the first- and second-order Bayliss-Turkel RBCs has been implemented in a finite element code. Preliminary results indicate that, compared with the use of an RBC on a circular domain, the new RBCs do significantly reduce the time needed to compute an approximate solution without sacrificing too much accuracy [12].

Further analysis of the properties of the different RBCs will hopefully provide guidance in the selection of an appropriate artificial boundary and radiation boundary condition. The experience gained from this analysis will be applied to the search for effective local approximate RBCs for other problems, including the full three-dimensional vector Helmholtz equation.

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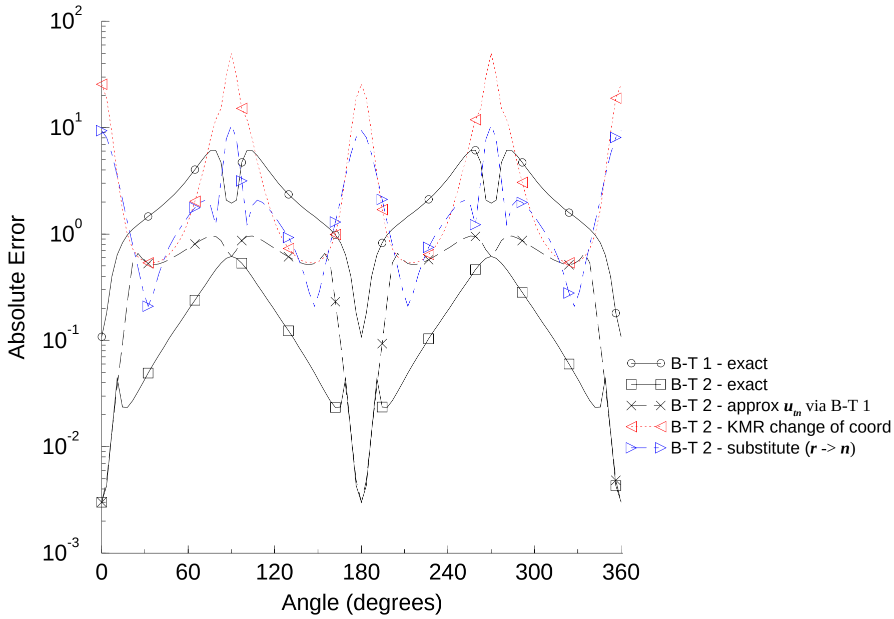
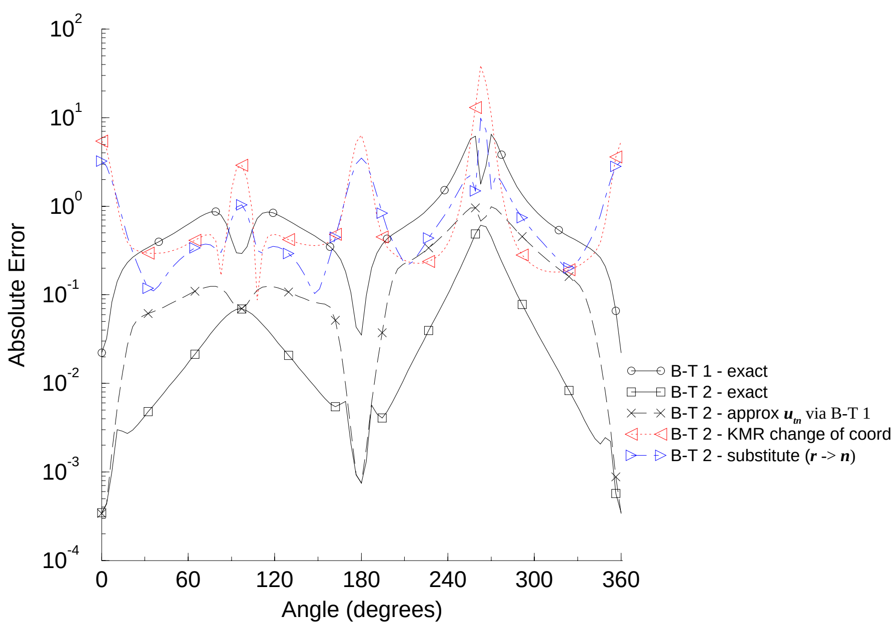
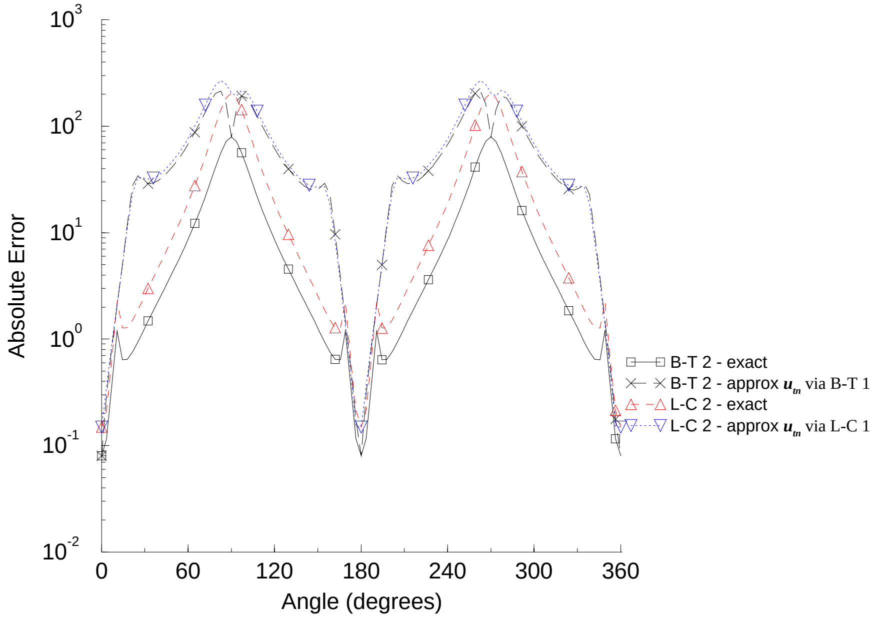


(a) $5\lambda \times 1\lambda$ *ellipse*

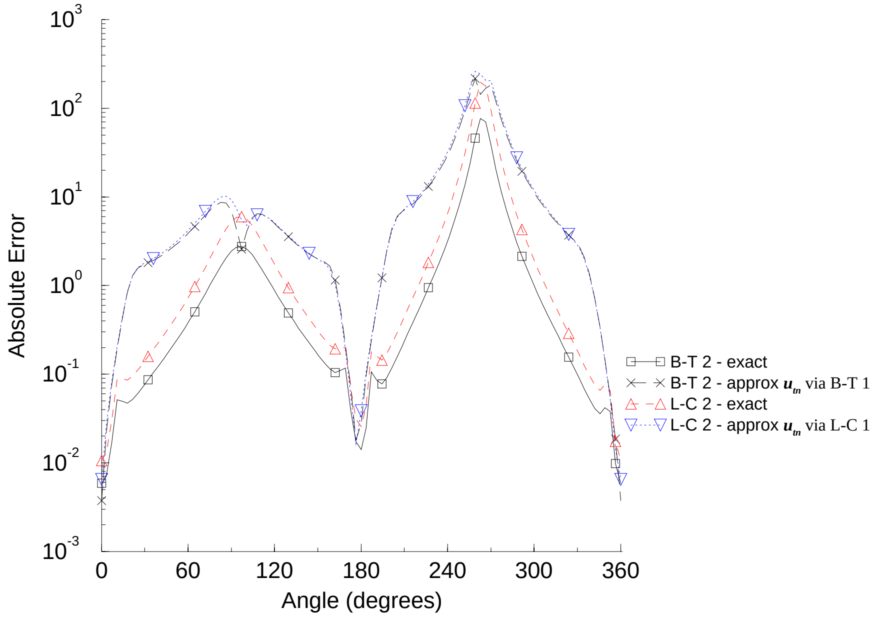


(b) $10\lambda \times 2\lambda$ *ellipse with center* $(1\lambda, 1\lambda)$

FIGURE 2. *Comparison of approximate RBCs demonstrating the importance of using an exact coordinate transformation. The absolute error,* $|\mathcal{B}U_0|$*, is computed along two different elliptical boundaries. In (a) the radii are* $a = 5\lambda$ *and* $b = 1\lambda$ *with center* $(0,0)$*; in (b) the radii are* $a = 10\lambda$ *and* $b = 2\lambda$ *with center* $(1\lambda, 1\lambda)$*.*



(a) $5\lambda \times 1\lambda$ ellipse with center $(0, 0)$



(b) $10\lambda \times 2\lambda$ ellipse with center $(1\lambda, 1\lambda)$

FIGURE 3. *Comparison of the approximate RBCs obtained by various combinations of the Bayliss-Turkel and Li-Cendes RBCs. The absolute error, $|\mathcal{B}U_2|$, is computed along two different elliptical boundaries: in (a) the radii are $a = 5\lambda$ and $b = 1\lambda$ with center $(0, 0)$; in (b) the radii are $a = 10\lambda$ and $b = 2\lambda$ with center $(1\lambda, 1\lambda)$.*