The ODE of World-Class Sprints

Step I – Understanding the Model

According to Keller's theory of competitive running, published in 1973, track sprints of up to 300 meters can be described by the following differential equation

$$\frac{dv}{dt} = A - \frac{v}{\tau}. \tag{1}$$

At the time of this analysis, the best constants were $A = 12.2\text{m/sec}^2$ and $\tau = 0.892\text{sec}$.

- What is an appropriate initial condition for the problem?
- In your own words, explain each of the terms in the differential equation, and how the terms combine to form a complete model. What is an interpretation of the two parameters A and τ? Does the differential equation seem reasonable in your experience? Do you know any other situations that could be described using this same model?

Step II – Finding and Interpreting Solutions

One reason for solving differential equations is to derive further consequences from the solution. The next set of questions explores the solution and its consequences.

- Solve the initial value problem for $v(t)$ symbolically in terms of the parameters A and τ.
- Find the distance $D(t)$ traveled in a sprint as a function of time symbolically in terms of A and τ.
- Find the acceleration function $a(t)$ as a function of time. Also, what differential equation does $a(t)$ satisfy? What are the initial conditions?
- Draw a graph of the distance, velocity, and acceleration over a reasonable time interval using the the 1973 parameters.
- When does the maximum acceleration occur for the 1973 parameters? How long does it take for the acceleration to drop to 10% of its maximum value?
- What is the runners maximum speed? How long does it take for the runner to reach 90% of this maximum. Compare the final speeds in a 100m and a 200m race. Explain your findings.
- Answer the previous two questions for general values of A and τ.

Step III – Extending the Problem (Data Fitting)

Another aspect of differential equations is to use the solution to develop the parameters of the problem. This is a mathematical form of the game *Jeopardy*: Given the solution (or the data), find the equation (or the parameters for the equation). In more advanced applications of differential equations, mathematicians and scientists call this an *inverse problem.* Inverse problems can be very difficult, and are an active area of research in applied mathematics.

Race officials often record the split times of runners in addition to the final times. Split times are the measure of time required to cover portions of the distance. In Table 1 are the results for the top four men and women in the 100m sprint at the 1993 World Track and Field Championships held in Stuttgart, Germany.

- To find out if athletic performance has changed since 1973, determine values for the parameters A and τ using the 1993 results.
- Compare the A and τ values of the men and women. Some track and field experts speculate that men's and women's abilities are becoming identical. Do your conclusions support this?

Step IV – Summary and Conclusion

The theoretical analysis and the work that you have done with the data should have given you a better insight into this model. To conclude the project, let's consider a few of the following points.

- Is it preferable to accelerate quickly at the start and then maintain the speed or to hold back and accelerate over a longer period of time? How would the model change to allow for weakening, i.e. deceleration, towards the end of the race? (This answer should be in the form of instructions, and explanation, to the sprinter and/or their coach.)
- Are the results produced by the model reasonable? Why does Keller suggest that this model is valid only for races up to 300m? Do you agree with this? What types of modifications would be needed for longer races (*e.g.*, 1600m, 10km, or marathon)?

Bibliography

[1] Alexandrov, I. and P. Lucht, Physics of sprinting, *American Jour. Physics*, **49**, 1977.

[2] Dunbar, Steven R., *The ODE of World-Class Sprints*, C-ODE-E Newsletter, Spring 1994, pp. 7–9.

[3] Keller, J.B., A theory of competitive running, *Physics Today*, Sept. 1973, p. 43.

Name	Nation	30 meters	60 meters	80 meters	100 meters
Linford Christie	GBR	3.85	6.45	8.15	9.87
Andre Cason	USA	3.83	6.43	8.15	9.92
Dennis Mitchell	USA	3.82	6.46	8.22	9.99
Carl Lewis	USA	3.95	6.59	8.30	10.02
Gail Devers	USA	4.09	6.95	8.86	10.82
Merlene Ottey	GHA	4.13	6.98	8.87	10.82
Gwen Torrence	USA	4.14	7.00	8.92	10.89
Irina Privalova	RUS	4.09	7.00	8.96	10.96

Table 1: Split times for 100m sprint, 1993 World Track and Field Championships. Data courtesy of Shawn Price of *Track and Field News*.

[4] Pritchard, W.G., Mathematical models of running, *SIAM Review*, Sept. 1992, **35**, pp. 359–379.

[5] Thompson, W.J., *Computing in Applied Science*, Wiley, 1984, pp. 107–109.