Phase Plane (§5.7)

PHASE-PLANE ANALYSIS (§5.7, p. 287)

```
> restart;
> with( plots ):
> with( DEtools ):
```

Initial value problems for nonlinear equations — both first-order systems and higher-order equations — are very difficult to solve in closed form. However, significant amounts of information can often be obtained directly from the equations. The phase portrait is a simple, yet powerful, tool for uncovering some of this information. This introduction to phase-plane analysis will involve the examination of Examples 3, 4, and 5 from Section 5.7 of Nagle and Saff. While the worksheet follows the same general outline as the text, the worksheet both includes additional information and omits other information. This worksheet is intended for parallel use with the text.

Objectives:

- understand what a phase portrait is
- be able to extract qualitative information from a phase portrait
- use the `DEplot` command to create phase portraits

New Maple Commands:

`scene=` – optional argument for `DEplot`, used to create phase portraits
`signum` – sign function

EXAMPLE 3

Determine the trajectories of the following system and discuss the stability of the critical points.

```
> SYS := [ diff( x(t), t ) = -y(t) * ( y(t) - 2 ),
>          diff( y(t), t ) = ( x(t) - 2 ) * ( y(t) - 2 ) ];
```

$$SYS := \left[\frac{\partial}{\partial t}\,\mathrm{x}(t) = -\mathrm{y}(t)\,(\mathrm{y}(t)-2),\ \frac{\partial}{\partial t}\,\mathrm{y}(t) = (\mathrm{x}(t)-2)\,(\mathrm{y}(t)-2)\right]$$

The critical points of this system were found, in Example 2, to be the single point $(2,0)$ and the line $y=2$. The trajectories for this system are the general solutions to

```
> ODE := diff( y(x), x ) = (x-2)/y(x);
```

$$ODE := \frac{\partial}{\partial x}\,\mathrm{y}(x) = \frac{x-2}{\mathrm{y}(x)}$$

This equation, which is separable, is easy to solve by hand or with Maple:

```
> dsolve( ODE, y(x) );
```

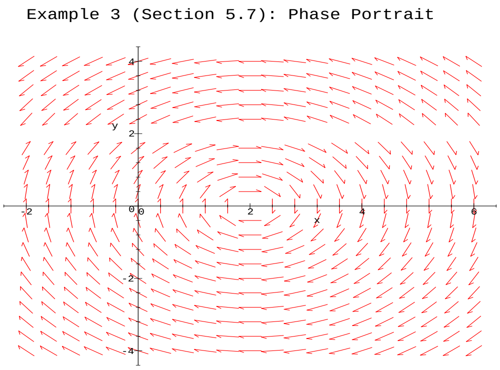
$$\frac{1}{2}\mathrm{y}(x)^2 - \frac{1}{2}x^2 + 2x = \_C1$$

Do you see how this solution is equivalent to the one given in the text?

The stability of the critical points is determined by the flow along each trajectory. One way in which this can be done is to create a "phase portrait" for this system. The phase portrait of a system is closely related to the direction field of a single equation – it shows how a particle moves from any point in space. The connection between direction fields and phase portraits is so close that the same Maple command is used for both: `DEplot` (from the `DEtools` package).

NOTE: This plot may take several minutes to appear; a more efficient method will be presented next.

```
> DEplot( SYS, [x,y], t=0..1, x=-2..6, y=-4..4, arrows=THIN, scene=[x,y],
>         title=`Example 3 (Section 5.7): Phase Portrait` );
```



The key to speeding up the computation of the phase portrait is to note that the system is autonomous (the right-hand side of the equations does not explicitly involve the independent variable). While Maple cannot detect this for itself, it is possible to provide the arguments in a form that allows Maple to take advantage of this property of the system. The preferred syntax for the creation of phase portraits for autonomous systems is to specify, as a list, the right-hand sides of the differential equations (without specific mention of the independent variable and in the same order as the dependent variables are listed in the second argument). For example:

```
> SYSa := [ -y*(y-2), (x-2)*(y-2) ];
```

$$SYSa := [-y\,(y-2), (x-2)\,(y-2)]$$

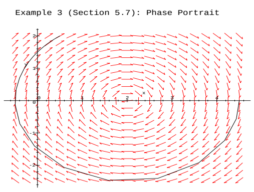
```
> DEplot( SYSa, [x,y], t=0..5, x=-2..6, y=-4..4, arrows=THIN, scene=[x,y],
>         title=`Example 3 (Section 5.7): Phase Portrait` );
```

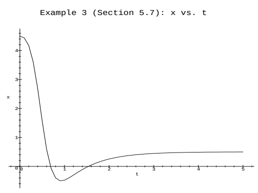
Understanding the information presented in a phase portrait is a little confusing — until you know what to read the graph. The discussion in the text should help with this. It also helps to be able to explicitly see the dependent variables plotted against time. The `scene=` argument of the `DEplot` command can be used to produce the plot of any combination of two or three variables for any initial condition. To illustrate, create the phase portrait and plot each component of the solution to this system with initial conditions $x(0) = 4.5$, $y(0) = 0$.

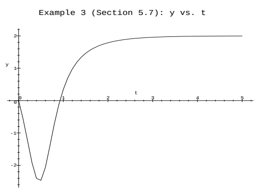
```
> IC := { [0,4.5,0] };
```

$$IC := \{[0, 4.5, 0]\}$$

```
> DEplot( SYSa, [x,y], t=0..5, IC, scene=[x,y], arrows=THIN, stepsize=0.1,
>         title=`Example 3 (Section 5.7): Phase Portrait` );
> DEplot( SYSa, [x,y], t=0..5, IC, scene=[t,x], arrows=THIN, stepsize=0.1,
>         title=`Example 3 (Section 5.7): x vs. t` );
> DEplot( SYSa, [x,y], t=0..5, IC, scene=[t,y], arrows=THIN, stepsize=0.1,
>         title=`Example 3 (Section 5.7): y vs. t` );
```







Notice how, during the first stage, the solutions for both x and y are decreasing with $x > 0$ and $y < 0$ – this corresponds to the quadrant of the phase plane with $x > 2$ and $y < 0$. The next stage of the solution takes place for $x < 2$ and $y < 0$; here x is decreasing and y is increasing. When $y > 0$, x increases. The last stage of the solution is its approach to one of the critical points on the line $y = 2$.

How would the phase portrait and trajectories look for the solution starting with $x(0) = 3.5$, $y(0) = 0$? $x(0) = 3.5$, $y(0) = 3$?

EXAMPLE 4

Use a software package to construct a phase plane diagram about the critical point $(0, 0)$ for the linear system (given, here, in a format suitable for `DEplot`)

```
> SYS := [ -2*x + y, -5*x - 4*y ];
```

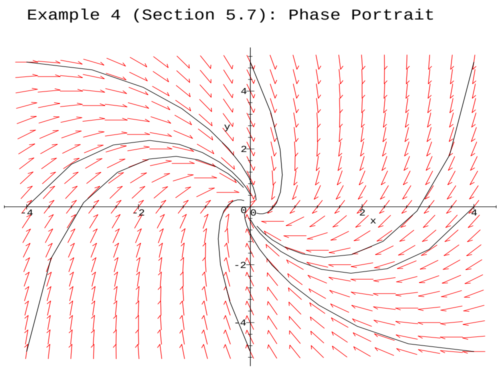
$$SYS := [-2\,x + y, -5\,x - 4\,y]$$

The discussion in the text begins with a view of the phase portrait for a variety of initial conditions on the rectangle $-4 \le x \le 4$, $-5 \le x \le 5$

```
> IC := { [0, 4, -5], [0, 4, 0], [0, 4, 5], [0, -4, 5],
>           [0, -4, -5], [0, -4, 0], [0, 0, -5], [0,0,5] };
```

$$IC := \{[0, 4, -5], [0, -4, -5], [0, -4, 5], [0, 4, 5], [0, -4, 0], [0, 4, 0], [0, 0, 5], [0, 0, -5]\}$$

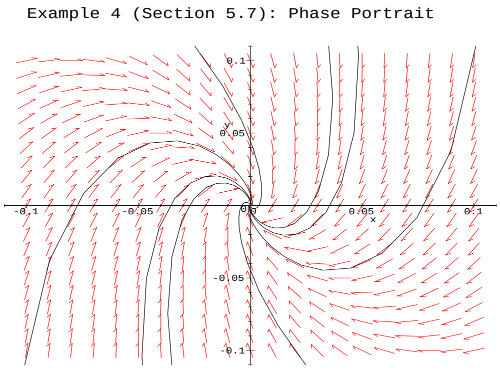
```
> DEplot( SYS, [ x, y ], t=0..1, IC, scene=[x,y], arrows=THIN, stepsize=0.1,
>          title=`Example 4 (Section 5.7): Phase Portrait` );
```



The trajectories, in phase space, appear to spiral towards the origin. How would the trajectories look when plotted against time? (Use Maple to check your intuition.)

To zoom in on the behavior close to the origin, all that is needed is to restrict the view to a rectangle, say, $-0.1 \leq x \leq 0.1$, $-0.1 \leq y \leq 0.1$. (Note that it is also necessary to increase the time interval.)

```
> DEplot( SYS, [ x, y ], t=0..3, IC, scene=[x,y], x=-0.1..0.1, y=-0.1..0.1,
>          arrows=THIN, stepsize=0.1,
>          title=`Example 4 (Section 5.7): Phase Portrait` );
```



This process can be continued (almost) indefinitely.

EXAMPLE 5

Describe the motion of a spring–mass system with "sticky friction" (see Nagle and Saff, p. 294) modelled by

$$m\frac{d^2x}{dt^2} + \mu \mathrm{sign}\left(\frac{dx}{dt}\right) + kx = 0$$

in the case where $m = \mu = k = 0$.

While `DEplot` will accept second- (and higher-)order equations in the first argument, it's not possible to generate a phase portrait in these cases. The first argument must correspond with a first-order system. In this example this is accomplished by introducing $v = \frac{dx}{dt}$ as the second dependent variable. Thus, the problem can be written ass

```
> SYS := [ v, -x - signum(v) ];
```

$$SYS := [\,v, -x - \mathrm{signum}(\,v\,)\,]$$

The phase portrait, together with the trajectories corresponding to the initial conditions

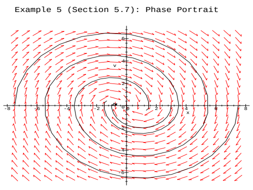
```
> IC := { [ 0, 7.5, 0 ], [ 0, -7.5, 0 ], [ 0, 3, 0 ] };
```

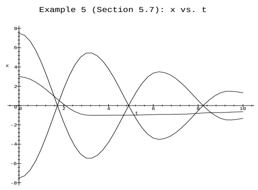
$$IC := \{\,[\,0, 7.5, 0\,], [\,0, -7.5, 0\,], [\,0, 3, 0\,]\,\}$$

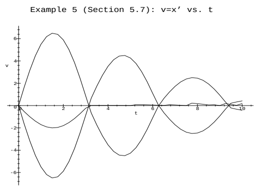
, can be produced using

```
> DEplot( SYS, [x,v], t=0..10, IC, arrows=THIN, stepsize=0.25,
>         title=`Example 5 (Section 5.7): Phase Portrait` );
> DEplot( SYS, [x,v], t=0..10, IC, scene=[t,x], arrows=THIN, stepsize=0.25,
>         title=`Example 5 (Section 5.7): x vs. t` );
```

```
> DEplot( SYS, [x,v], t=0..10, IC, scene=[t,v], arrows=THIN, stepsize=0.25,
>         title=`Example 5 (Section 5.7): v=x' vs. t` );
```







For a full description of these solutions, please consult Nagle and Saff.

To conclude, it is possible to produce a three-dimensional plot that, when viewed with different orientations, can display either the phase plane or either of the two solution spaces.

```
> DEplot( SYS, [x,v], t=0..10, IC, scene=[t,x,v], stepsize=0.25,
>         title=`Example 5 (Section 5.7): 3D View of Trajectories` );
```