Laplace Transforms (Ch. 7)

LAPLACE TRANSFORMS (Ch. 7)

```
> restart;
> with( plots ):
> with( DEtools ):
```

The Laplace transform is a very common, and useful, technique for solving and analyzing the solution of initial value problems for linear and constant coefficient equations and systems. The key property of the Laplace transform is that derivatives are transformed into powers; thus, the differential equation is transformed into an algebraic equation. The focus of this worksheet is to understand the Laplace transform, and the inverse Laplace transform, more than the solution of a differential equation.

Objectives:

- to understand the definition of the Laplace transform and the inverse Laplace transform
- to appreciate the general applicability of Laplace transforms to the solution of linear ODEs (both single equations and systems)
- to understand and be able to work with piecewise defined functions
- to become familiar with several of the different ways in which Maple can deal with Laplace transforms

New Maple Commands:

`laplace`	– apply the Laplace transform
`invlaplace`	– apply the inverse Laplace transform
`convert(..., parfrac, s)`	– compute the partial fraction decomposition of an expression
`completesquare`	– complete the square of a polynomial of degree 2

The full implementation of the Laplace transform is available only after the `laplace` library is read. Likewise, the `student` package is needed if it is desired to complete the square on an expression.

```
> readlib( laplace ):
> with( student );
```

$[D, \mathit{Doubleint}, \mathit{Int}, \mathit{Limit}, \mathit{Lineint}, \mathit{Sum}, \mathit{Tripleint}, \mathit{changevar}, \mathit{combine},$
$\quad \mathit{completesquare}, \mathit{distance}, \mathit{equate}, \mathit{extrema}, \mathit{integrand}, \mathit{intercept}, \mathit{intparts},$

isolate, *leftbox*, *leftsum*, *makeproc*, *maximize*, *middlebox*, *middlesum*, *midpoint*, *minimize*, *powsubs*, *rightbox*, *rightsum*, *showtangent*, *simpson*, *slope*, *trapezoid*, *value*]

INTRODUCTION TO LAPLACE TRANSFORMS

The Laplace transform is defined in terms of an improper integral. For example,

```
> int( 1*exp(-s*t), t=0..infinity );
```

$$\lim_{t\to\infty-} -\frac{e^{(-s\,t)}}{s}+\frac{1}{s}$$

```
> int( exp(a*t)*exp(-s*t), t=0..infinity );
```

$$\lim_{t\to\infty-} -\frac{e^{(a\,t)}\,e^{(-s\,t)}}{-a+s}+\frac{1}{-a+s}$$

```
> int( cos(a*t)*exp(-s*t), t=0..infinity );
```

$$\lim_{t\to\infty-} -\frac{s\,e^{(-s\,t)}\cos(a\,t)}{s^2+a^2}+\frac{a\,e^{(-s\,t)}\sin(a\,t)}{s^2+a^2}+\frac{s}{s^2+a^2}$$

Notice that these are not the expected results. In each case, the anticipated result is present, but so is a limit. This limit arises, of course, because the integral is improper.

Questions

- What do we know about s that Maple does not know?
- How does this knowledge allow us to conclude that each limit is zero?

The `laplace` command does know the necessary information about the transform parameter. Here are the same problems using `laplace`:

```
> laplace( 1, t, s );
```

$$\frac{1}{s}$$

```
> laplace( exp(a*t), t, s );
```

$$\frac{1}{-a+s}$$

```
> laplace( cos(a*t), t, s );
```

$$\frac{s}{s^2+a^2}$$

Inverse Laplace transforms work similarly, except that there is no explicit formula. Regardless, it is still quite beneficial to use Maple to convert an expression into a form that can be inverted using the table of Laplace Transforms (inside back cover of Nagle and Saff). The `convert(..., parfrac)` and `completesquare` commands are particularly helpful.

For example, the function

```
> F1 := (7*s-1)/(s^3-7*s-6);
```

$$F1 := \frac{7\,s-1}{s^3-7\,s-6}$$

obviously has at least one real root. The partial fraction decomposition is

```
> convert( F1, parfrac, s );
```

$$\frac{1}{s-3} - 3\,\frac{1}{s+2} + 2\,\frac{1}{s+1}$$

The inverse Laplace transform is now easy to identify. Check your answer with the answer provided by Maple:

```
> f1 := invlaplace( F1, s, t );
```

$$f1 := \mathrm{e}^{(3\,t)} - 3\,\mathrm{e}^{(-2\,t)} + 2\,\mathrm{e}^{(-t)}$$

The next example

```
> F2 := (s-1)/(s^2-2*s+5);
```

$$F2 := \frac{s-1}{s^2-2\,s+5}$$

involves an irreducible quadratic term, which should be rewritten in a form with a complete square. This is achieved by using the `completesqare` command:

```
> completesquare( F2 );
```

$$\frac{s-1}{(\,s-1\,)^2+4}$$

The presence of $(s-1)$ says that the inverse Laplace transform includes e^t. The inverse Laplace transform of $\frac{s}{s^2+4}$ is $\cos(2t)$; thus, the inverse Laplace transform of `F` must be $f(t) = e^t\cos(2t)$.œAs before, this answer is easily checked using Maple

```
> f2 := invlaplace( F2, s, t );
```

$$f2 := \mathrm{e}^t \cos(\,2\,t\,)$$

The third example looks innocuous enough, but the exponential term does not appear in any of the transforms first presented. Even though you may not yet possess enough knowledge to determine the inverse Laplace transform of this function, we can still ask Maple for an answer.

```
> F3 := exp(-2*s)/s^3;
```

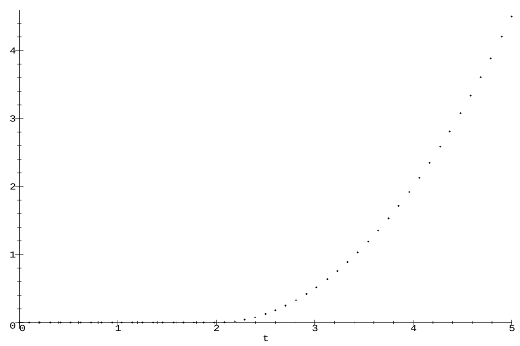
$$F3 := \frac{\mathrm{e}^{(-2\,s)}}{s^3}$$

```
> f3 := invlaplace( F3, s, t );
```

$$f3 := \frac{1}{2}\,\mathrm{Heaviside}(\,t-2\,)\,(\,t-2\,)^2$$

What is this function `Heaviside`? One of the easiest ways to learn about Heaviside functions is to plot this function, and attempt to understand how it relates to this function.

```
> plot( f3, t=0..5,
>       title=`Function with Heaviside` );
```



Notice that the solution is identically zero for $t < 2$, and is quadratic with a vertex at $t = 2$ for $t > 2$. This suggests that the Heaviside function is zero when its argument is negative and is identically one when the argument is positive. (How might you use Maple to test this conjecture?)

LAPLACE TRANSFORMS FOR INITIAL VALUE PROBLEMS

Example 1

These are the problems that are of most interest. To illustrate the general procedure, consider the problem of solving the initial value problem

```
> ODE := diff( y(t), t$2 ) - 3*diff( y(t), t ) + 2*y(t)
>        = cos(t) + (t^2-3)*exp(-2*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\,\mathrm{y}(t)\right) - 3\left(\frac{\partial}{\partial t}\,\mathrm{y}(t)\right) + 2\,\mathrm{y}(t) = \cos(t) + (t^2 - 3)\,\mathrm{e}^{(-2\,t)}$$

```
> IC := y(0) = 0, D(y)(0) = 1;
```

$$IC := \mathrm{y}(0) = 0, \mathrm{D}(y)(0) = 1$$

using Laplace transforms.

The first step is to apply the Laplace transform to the entire equation

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := (\mathrm{laplace}(\mathrm{y}(t), t, s)\,s - \mathrm{y}(0))\,s - \mathrm{D}(y)(0) - 3\,\mathrm{laplace}(\mathrm{y}(t), t, s)\,s$$
$$+ 3\,\mathrm{y}(0) + 2\,\mathrm{laplace}(\mathrm{y}(t), t, s) = -\frac{2\,s^4 + 6\,s^3 + s^2 + 4\,s + 10}{(s^2 + 1)\,(s + 2)^3}$$

Note the way in which the Laplace transform of the solution is denoted in this equation. Since specific initial conditions are known, these should be introduced at the earliest convenience. For example:

```
> ODE_lap2 := subs( IC, ODE_lap );
```

$$ODE\_lap2 := \mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,)\, s^2 - 1 - 3\,\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,)\, s + 2\,\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = -\frac{2\,s^4 + 6\,s^3 + s^2 + 4\,s + 10}{(\,s^2+1\,)\,(\,s+2\,)^3}$$

Next, it is necessary to explicitly solve for the Laplace transform of the solution:

```
> SOLN_lap := solve( ODE_lap2, { laplace( y(t), t, s ) } );
```

$$SOLN\_lap := \left\{\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = -\left(-1 + 2\,\frac{s^4}{(\,s^2+1\,)\,(\,s+2\,)^3} + 6\,\frac{s^3}{(\,s^2+1\,)\,(\,s+2\,)^3} + \frac{s^2}{(\,s^2+1\,)\,(\,s+2\,)^3} + 4\,\frac{s}{(\,s^2+1\,)\,(\,s+2\,)^3} + 10\,\frac{1}{(\,s^2+1\,)\,(\,s+2\,)^3}\right) \Big/ (\,s^2 - 3\,s + 2\,)\right\}$$

The solution can be obtained in one step using `invlaplace`, but the solution is also available by inspection once the right-hand side is written in its partial fraction decomposition.

```
> SOLN_lap2 := convert( SOLN_lap, parfrac, s );
```

$$SOLN\_lap2 := \left\{\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = \frac{1}{6}\,\frac{1}{(\,s+2\,)^3} + \frac{7}{72}\,\frac{1}{(\,s+2\,)^2} - \frac{179}{864}\,\frac{1}{s+2} - \frac{31}{54}\,\frac{1}{s-1} + \frac{109}{160}\,\frac{1}{s-2} + \frac{1}{10}\,\frac{s-3}{s^2+1}\right\}$$

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \left\{\mathrm{y}(\,t\,) = \frac{1}{12}\,t^2\,\mathrm{e}^{(\,-2\,t\,)} + \frac{7}{72}\,t\,\mathrm{e}^{(\,-2\,t\,)} - \frac{179}{864}\,\mathrm{e}^{(\,-2\,t\,)} - \frac{31}{54}\,\mathrm{e}^{t} + \frac{109}{160}\,\mathrm{e}^{(\,2\,t\,)} + \frac{1}{10}\cos(\,t\,) - \frac{3}{10}\sin(\,t\,)\right\}$$

Compare this solution with the solutions obtained by directly inverting the Laplace transform once it is available

```
> invlaplace( SOLN_lap, s, t );
```

$$\left\{\mathrm{y}(\,t\,) = \frac{1}{12}\,t^2\,\mathrm{e}^{(\,-2\,t\,)} + \frac{7}{72}\,t\,\mathrm{e}^{(\,-2\,t\,)} - \frac{179}{864}\,\mathrm{e}^{(\,-2\,t\,)} - \frac{31}{54}\,\mathrm{e}^{t} + \frac{109}{160}\,\mathrm{e}^{(\,2\,t\,)} + \frac{1}{10}\cos(\,t\,) - \frac{3}{10}\sin(\,t\,)\right\}$$

or the solution obtained by explicitly telling Maple to use Laplace transforms to solve the problem.

```
> dsolve( { ODE, IC }, y(t), method=laplace );
```

$$\mathrm{y}(\,t\,) = \frac{1}{12}\,t^2\,\mathrm{e}^{(\,-2\,t\,)} + \frac{7}{72}\,t\,\mathrm{e}^{(\,-2\,t\,)} - \frac{179}{864}\,\mathrm{e}^{(\,-2\,t\,)} - \frac{31}{54}\,\mathrm{e}^{t} + \frac{109}{160}\,\mathrm{e}^{(\,2\,t\,)} + \frac{1}{10}\cos(\,t\,) - \frac{3}{10}\sin(\,t\,)$$

Example 2

For a second example consider the problem of finding the general solution to

```
> ODE := diff( y(t), t$2 )  + 2*y(t) = cos(2*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\,\mathrm{y}(\,t\,)\right) + 2\,\mathrm{y}(\,t\,) = \cos(\,2\,t\,)$$

The Laplace transform of this equation is

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := (\,\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,)\, s - \mathrm{y}(\,0\,)\,)\, s - \mathrm{D}(\,y\,)(\,0\,) + 2\,\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = \frac{s}{s^2+4}$$

```
> SOLN_lap := solve( ODE_lap, { laplace(y(t),t,s) } );
```

$$SOLN\_lap := \left\{ \mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = -\frac{-s\,\mathrm{y}(\,0\,) - \mathrm{D}(\,y\,)(\,0\,) - \dfrac{s}{s^2+4}}{s^2+2} \right\}$$

```
> invlaplace( SOLN_lap, s, t );
```

$$\left\{ \mathrm{y}(\,t\,) = -\frac{1}{2}\cos(\,2\,t\,) + \frac{1}{2}\mathrm{D}(\,y\,)(\,0\,)\sin\left(\sqrt{2}\,t\right)\sqrt{2} + \frac{1}{2}\cos\left(\sqrt{2}\,t\right) + \mathrm{y}(\,0\,)\cos\left(\sqrt{2}\,t\right) \right\}$$

Note the explicit use of the initial conditions in this solution.

As before, there are several alternate methods to reach the same solution, including

```
> dsolve( ODE, y(t), method=laplace );
```

$$\mathrm{y}(\,t\,) = -\frac{1}{2}\cos(\,2\,t\,) + \frac{1}{2}\mathrm{D}(\,y\,)(\,0\,)\sin\left(\sqrt{2}\,t\right)\sqrt{2} + \frac{1}{2}\cos\left(\sqrt{2}\,t\right) + \mathrm{y}(\,0\,)\cos\left(\sqrt{2}\,t\right)$$

It is particularly noteworthy to compare this solution with the one returned by `dsolve` when the solution method is not explicitly specified:

```
> dsolve( ODE, y(t) );
```

$$\begin{aligned}\mathrm{y}(\,t\,) = {} & \frac{1}{4}\cos\left(\sqrt{2}\,t\right)\cos\left(2\,t+\sqrt{2}\,t\right)\sqrt{2} - \frac{1}{4}\cos\left(\sqrt{2}\,t\right)\cos\left(2\,t+\sqrt{2}\,t\right)\\ & - \frac{1}{4}\cos\left(\sqrt{2}\,t\right)\cos\left(-2\,t+\sqrt{2}\,t\right)\sqrt{2} - \frac{1}{4}\cos\left(\sqrt{2}\,t\right)\cos\left(-2\,t+\sqrt{2}\,t\right)\\ & - \frac{1}{4}\sin\left(\sqrt{2}\,t\right)\sin\left(-2\,t+\sqrt{2}\,t\right)\sqrt{2} - \frac{1}{4}\sin\left(\sqrt{2}\,t\right)\sin\left(-2\,t+\sqrt{2}\,t\right)\\ & + \frac{1}{4}\sin\left(\sqrt{2}\,t\right)\sin\left(2\,t+\sqrt{2}\,t\right)\sqrt{2} - \frac{1}{4}\sin\left(\sqrt{2}\,t\right)\sin\left(2\,t+\sqrt{2}\,t\right) + \_C1\cos\left(\sqrt{2}\,t\right)\\ & + \_C2\sin\left(\sqrt{2}\,t\right)\end{aligned}$$

This type of solution has been observed previously (see the `forced.ms`). The key in that situation was to ask Maple to apply a variety of trigonometric identities to convert the solution to a more standard form:

```
> combine( ", trig );
```

$$\mathrm{y}(\,t\,) = -\frac{1}{2}\cos(\,2\,t\,) + \_C1\cos\left(\sqrt{2}\,t\right) + \_C2\sin\left(\sqrt{2}\,t\right)$$

Example 3

The same techniques can be used to solve problems that involve piecewise-defined forcing functions (`Heaviside`) and/or impulse functions (`Dirac`). For example, the general solution to the equation

```
> ODE := diff( y(t), t$2 ) + 2*diff( y(t), t )
>        = 3*Dirac(t-1) + Heaviside(t-Pi)*sin(t-Pi) + exp(-t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\mathrm{y}(\,t\,)\right) + 2\left(\frac{\partial}{\partial t}\mathrm{y}(\,t\,)\right) = 3\,\mathrm{Dirac}(\,t-1\,) - \mathrm{Heaviside}(\,t-\pi\,)\sin(\,t\,) + \mathrm{e}^{(\,-t\,)}$$

can be found using the Laplace transform, exactly as before (this should illustrate the general structure of these problems):

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := (\mathrm{laplace}(\mathrm{y}(t), t, s)\, s - \mathrm{y}(0))\, s - \mathrm{D}(y)(0) + 2\,\mathrm{laplace}(\mathrm{y}(t), t, s)\, s - 2\,\mathrm{y}(0) = 3\,\mathrm{e}^{(-s)} + \frac{\mathrm{e}^{(-\pi s)}}{s^2+1} + \frac{1}{s+1}$$

```
> SOLN_lap := solve( ODE_lap, { laplace( y(t), t, s ) } );
```

$$SOLN\_lap := \left\{\mathrm{laplace}(\mathrm{y}(t), t, s) = -\frac{-s\,\mathrm{y}(0) - \mathrm{D}(y)(0) - 2\,\mathrm{y}(0) - 3\,\mathrm{e}^{(-s)} - \dfrac{\mathrm{e}^{(-\pi s)}}{s^2+1} - \dfrac{1}{s+1}}{s^2 + 2\,s}\right\}$$

```
> SOLN := invlaplace( SOLN_lap, s, t );
```

$$SOLN := \left\{\mathrm{y}(t) = \mathrm{y}(0)\,\mathrm{e}^{(-2\,t)} + \mathrm{D}(y)(0)\left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{(-2\,t)}\right) + 2\,\mathrm{y}(0)\left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{(-2\,t)}\right) + 3\,\mathrm{Heaviside}(t-1)\left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{(-2\,t+2)}\right) + \mathrm{Heaviside}(t-\pi)\left(\frac{1}{10}\,\frac{-5\cos(t-\pi)\,(\mathrm{e}^{(t-\pi)})^2 + \mathrm{e}^{(2\,t-2\,\pi)}\cos(t-\pi) - 2\,\mathrm{e}^{(2\,t-2\,\pi)}\sin(t-\pi)}{(\mathrm{e}^{(t-\pi)})^2} + \frac{1}{10}\,\frac{5\,(\mathrm{e}^{(t-\pi)})^2 - 1}{(\mathrm{e}^{(t-\pi)})^2}\right) - \frac{1}{\mathrm{e}^t} + \frac{1}{2}\,\frac{(\mathrm{e}^t)^2+1}{(\mathrm{e}^t)^2}\right\}$$

This solution look needlessly complicated (see, e.g., the common terms $e^{2t-2\pi}$ in the numerator and denominator of the coefficient of `Heaviside(t-Pi)`. The solution can be simplified

```
> SOLN := map( simplify, op(SOLN) );
```

$$SOLN := \mathrm{y}(t) = \frac{1}{2}\mathrm{D}(y)(0) - \frac{1}{2}\mathrm{e}^{(-2\,t)}\,\mathrm{D}(y)(0) + \mathrm{y}(0) + \frac{3}{2}\mathrm{Heaviside}(t-1) - \frac{3}{2}\mathrm{e}^{(-2\,t+2)}\,\mathrm{Heaviside}(t-1) + \frac{2}{5}\mathrm{Heaviside}(t-\pi)\cos(t) + \frac{1}{5}\mathrm{Heaviside}(t-\pi)\sin(t) + \frac{1}{2}\mathrm{Heaviside}(t-\pi) - \frac{1}{10}\mathrm{e}^{(-2\,t+2\,\pi)}\,\mathrm{Heaviside}(t-\pi) - \mathrm{e}^{(-t)} + \frac{1}{2} + \frac{1}{2}\mathrm{e}^{(-2\,t)}$$

and written in a format that clearly identifies the role of the initial data and the different solutions on each interval

```
> SOLN := collect( SOLN, { y(0), D(y)(0), Heaviside(t-1), Heaviside(t-Pi) } );
```

$$SOLN := \mathrm{y}(t) = \mathrm{y}(0) + \mathrm{D}(y)(0)\left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{(-2\,t)}\right) + \left(\frac{3}{2} - \frac{3}{2}\mathrm{e}^{(-2\,t+2)}\right)\mathrm{Heaviside}(t-1) + \left(\frac{1}{2} - \frac{1}{10}\mathrm{e}^{(-2\,t+2\,\pi)} + \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t)\right)\mathrm{Heaviside}(t-\pi) + \frac{1}{2}\mathrm{e}^{(-2\,t)} - \mathrm{e}^{(-t)} + \frac{1}{2}$$

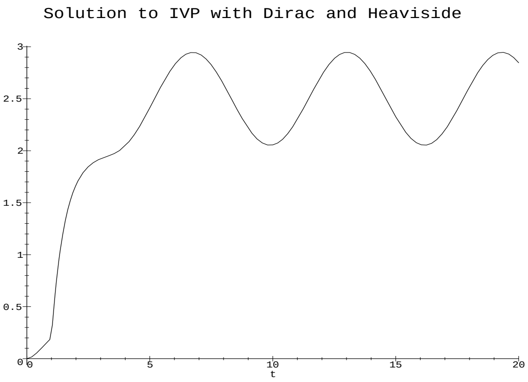
A plot of the solution with initial conditions

```
> IC := y(0)=0, D(y)(0)=0;
```

$$IC := \mathrm{y}(0) = 0, \mathrm{D}(y)(0) = 0$$

is also illuminating

```
> plot( subs( IC, rhs(SOLN) ), t=0..20,
>       title=`Solution to IVP with Dirac and Heaviside` );
```



The same solution can be obtained directly using `dsolve` with `method=laplace`.

LAPLACE TRANSFORMS FOR SYSTEMS OF LINEAR ODES

One of the great strengths of the Laplace transform is its applicability to both single equations and systems using exactly the same procedure; the only change is that now the algebraic equations will be a linear system. We illustrate with a single example.

```
> SYS := { diff( x(t), t ) = y(t) + sin(t),
>          diff( y(t), t ) = x(t) + cos(t) };
```

$$SYS := \left\{ \frac{\partial}{\partial t}\mathrm{y}(t) = \mathrm{x}(t) + \cos(t), \frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t) + \sin(t) \right\}$$

```
> SYS_lap := laplace( SYS, t, s );
```

$$SYS\_lap := \Big\{ \mathrm{laplace}(\mathrm{y}(t), t, s)\, s - \mathrm{y}(0) = \mathrm{laplace}(\mathrm{x}(t), t, s) + \frac{s}{s^2+1},$$
$$\mathrm{laplace}(\mathrm{x}(t), t, s)\, s - \mathrm{x}(0) = \mathrm{laplace}(\mathrm{y}(t), t, s) + \frac{1}{s^2+1} \Big\}$$

```
> SOLN_lap := solve( SYS_lap,
>                  { laplace( x(t), t, s ), laplace( y(t), t, s ) } );
```

$$SOLN\_lap := \Big\{ \mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{s\,\mathrm{y}(0) + \mathrm{x}(0) + 1}{s^2 - 1},$$
$$\mathrm{laplace}(\mathrm{x}(t), t, s) = \frac{\mathrm{y}(0)\, s^2 + \mathrm{x}(0)\, s^3 + s\,\mathrm{x}(0) + 2\, s + \mathrm{y}(0)}{(s^2 - 1)(s^2 + 1)} \Big\}$$

```
> SOLN_lap2 := convert( ", parfrac, s );
```

$$SOLN\_lap2 := \left\{\mathrm{laplace}(\,\mathrm{y}(\,t\,), t, s\,) = \frac{1}{2}\,\frac{\mathrm{y}(\,0\,) + \mathrm{x}(\,0\,) + 1}{s-1} + \frac{1}{2}\,\frac{\mathrm{y}(\,0\,) - \mathrm{x}(\,0\,) - 1}{s+1},\right.$$
$$\left.\mathrm{laplace}(\,\mathrm{x}(\,t\,), t, s\,) = \frac{1}{2}\,\frac{\mathrm{y}(\,0\,) + \mathrm{x}(\,0\,) + 1}{s-1} - \frac{1}{2}\,\frac{\mathrm{y}(\,0\,) - \mathrm{x}(\,0\,) - 1}{s+1} - \frac{s}{s^2+1}\right\}$$

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \Big\{$$
$$\mathrm{x}(\,t\,) = \frac{1}{2}\,\mathrm{y}(\,0\,)\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{x}(\,0\,)\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{e}^t - \frac{1}{2}\,\mathrm{y}(\,0\,)\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{x}(\,0\,)\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{e}^{(-t)} - \cos(\,t\,),$$
$$\left.\mathrm{y}(\,t\,) = \frac{1}{2}\,\mathrm{y}(\,0\,)\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{x}(\,0\,)\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{y}(\,0\,)\,\mathrm{e}^{(-t)} - \frac{1}{2}\,\mathrm{x}(\,0\,)\,\mathrm{e}^{(-t)} - \frac{1}{2}\,\mathrm{e}^{(-t)}\right\}$$

```
> SOLN2 := collect( ", { x(0), y(0) } );
```

$$SOLN2 := \left\{\mathrm{y}(\,t\,) = \left(\frac{1}{2}\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{e}^{(-t)}\right)\,\mathrm{y}(\,0\,) + \left(-\frac{1}{2}\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{e}^t\right)\,\mathrm{x}(\,0\,) - \frac{1}{2}\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{e}^t,\right.$$
$$\left.\mathrm{x}(\,t\,) = \left(-\frac{1}{2}\,\mathrm{e}^{(-t)} + \frac{1}{2}\,\mathrm{e}^t\right)\,\mathrm{y}(\,0\,) + \left(\frac{1}{2}\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{e}^{(-t)}\right)\,\mathrm{x}(\,0\,) - \cos(\,t\,) + \frac{1}{2}\,\mathrm{e}^t + \frac{1}{2}\,\mathrm{e}^{(-t)}\right\}$$