Hypergeometric Functions

Step I – Understanding the Problem

The *hypergeometric equation* is the linear second-order ODE

$$x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0$$

with parameters α, β, and γ. The Method of Frobenius yields the first solution to this equation (the *hypergeometric functions*):

$$F(\alpha,\beta,\gamma;x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n$$

where $(t)_n = \dfrac{\Gamma(t+n)}{\Gamma(t)}$.

A number properties of hypergeometric functions were established in the text [2, §8.9]. For example, $F(1,\beta,\beta;x)$ is a geometric series and $F(\frac{1}{2},1,\frac{3}{2};-x^2) = \frac{\arctan x}{x}$. In this project, we will develop additional properties of hypergeometric functions.

Benefits derived from the completion of this project will include improved series-manipulation skills and a enhanced knowledge of hypergeometric functions.

Step II – Basic Identities

Verify each of the following facts about hypergeometric functions:

- the power series for F is known to converge on $0 < x < 1$; find the largest set on which the series for F converges.
- determine conditions on α, β, and γ that guarantee that $F(\alpha,\beta,\gamma;x)$ is a polynomial (*i.e.*, when does the series involve only a finite number of non-zero terms?)
- verify that $\frac{d}{dx}F(\alpha,\beta,\gamma;x) = \frac{\alpha\beta}{\gamma}F(1+\alpha,1+\beta,1+\gamma;x)$
- express $\frac{d^2}{dx^2}F(alpha,\beta,\gamma;x)$ in terms of a single evaluation of F (for appropriate arguments) multiplied by an appropriate coefficient

Step III – Hypergeometric and Elementary Functions

Hypergeometric functions can frequently be used to provide a connection between a variety of elementary functions. A number of these relationships will be explored in this part of the project.

- show that $F(\alpha,1,\alpha;x) = F(1,1,1;x) = (1-x)^{-1}$.
- show that $F(-n,\beta,\beta;x) = (1+x)^{-n}$.
- show that $F(1/2,1/2,3/2;x^2) = \arcsin x$.

- show that $F(1-n,1,2;x) = \frac{1-(1-x)^n}{nx}$.
- express $F(\alpha,\beta,\gamma;1)$ as a ratio of two products involving the gamma function, Γ.

Step IV – Hypergeometric Functions and Differential Equations

The hypergeometric equation is a second-order linear ODE, so it should have two linearly independent solutions. When γ is not an integer, the Method of Frobenius yields a second solution

$$y_2(x) = x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x).$$

Show that $y_1(x) = F(\alpha,\beta,\gamma;x)$ and y_2 are linearly independent solution to the hypergeometric equation.

Derivatives of the hypergeometric function lead to several interesting identities. For example,

$$\frac{d}{dx}F(\alpha,\beta,\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x).$$

Verify this identity. Then, derive the corresponding identities for $\frac{d^2}{dx^2}F(\alpha,\beta,\gamma;x)$ and, for any integer $n \geq 1$, $\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x)$.

Step V – Summary and Conclusion

Note that MAPLE knows about the hypergeometric function. Actually, `hypergeom` is the generalized hypergeometric function. This function is similar to F, but has a few differences. See the on-line help, `?hypergeom`, for complete information about hypergeometric functions and MAPLE.

While the hypergeometric function is rather complicated, the fact that it is related to so many functions (logarithms, inverse tangent, Legendre polynomials, *etc.*) gives an indication of the breadth of problems in which hypergeometric functions can arise.

Bibliography

[1] Abrowitz M., I.A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, 1972.

[2] Nagle, R. Kent, Edward B. Saff, *Fundamentals of Differential Equations*, 4$^{\text{th}}$ edition, Addison–Wesley, 1996.