Forced Vibrations (§5.3)

FORCED VIBRATIONS (§5.3, p. 253)

```
> restart;
> with( plots ):
> with( DEtools ):
> constants := op( { constants } minus { gamma } ):
```

The solution to a second-order linear constant coefficient ODE with a trigonometric forcing function can be difficult to analyze by inspection. Several different characteristics of the solution are identified in this section, and this worksheet. The discussion in the text is valid for any problem of this type (i.e., the analysis is done for general parameters). The same analysis could be repeated using Maple. However, the general formulae are only a small part of the story; understanding the physical meaning of the different quantities that will be discussed is, in my opinion, of more long-lasting importance. Thus, the discussion found in this worksheet will only consider a specific set of parameters. The complementary emphases of the text and the worksheet should combine to create a more complete picture of this important material.

Objectives:

- introduce concept of frequency gain, frequency response curve, and resonance frequency
- emphasize physical understanding and computations from first principles over memorization of formulae
- apply and combine Maple techniques for solving second-order linear constant coefficient ODEs to extract information from the solution

New Maple Commands:

`coeff`	– extract the coefficient of a specific term of an expression
`collect`	– collect coefficients of specified terms in an expression
`combine(..., trig)`	– combines a product of trigonometric terms into a sum of trigonometric terms

Consider the second-order linear constant coefficient ODE

```
> ODE := diff( x(t), t,t ) + 2*diff( x(t), t ) + 5*x(t) = 3*cos(4*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)\right) + 2\left(\frac{\partial}{\partial t}\,\mathrm{x}(t)\right) + 5\,\mathrm{x}(t) = 3\cos(4\,t)$$

with homogeneous initial conditions

```
> IC1 := x(0)=0;
> IC2 := D(x)(0)=0;
```

$$IC1 := \mathrm{x}(0) = 0$$

$$IC2 := \mathrm{D}(x)(0) = 0$$

The equation of motion (i.e., the solution) to this IVP can be obtained directly using `dsolve`

```
> EOM := dsolve( { ODE, IC1, IC2 }, x(t) );
```

$$EOM := \mathrm{x}(t) = \frac{9}{74}\cos(2t)\cos(6t) - \frac{3}{148}\cos(2t)\sin(6t) - \frac{3}{5}\cos(2t)^2 + \frac{3}{10}\sin(2t)\cos(2t) + \frac{3}{10} + \frac{3}{148}\sin(2t)\cos(6t) + \frac{9}{74}\sin(2t)\sin(6t) + \frac{33}{185}\mathrm{e}^{(-t)}\cos(2t) - \frac{63}{370}\mathrm{e}^{(-t)}\sin(2t)$$

This solution is NOT what we would have expected.

Questions

1. What should the solution look like?
2. Why?
3. How did Maple come up with this solution?

There are trigonometric identities that combine a product of trigonometric terms to form a sum of trigonometric terms. The Maple command that applies these identities is `combine` (with an optional second argument of `trig`).

```
> SOLN := combine( rhs(EOM), trig );
```

$$SOLN := -\frac{33}{185}\cos(4t) + \frac{24}{185}\sin(4t) + \frac{33}{185}\mathrm{e}^{(-t)}\cos(2t) - \frac{63}{370}\mathrm{e}^{(-t)}\sin(2t)$$

Questions

- Identify the transient and steady-state solutions.

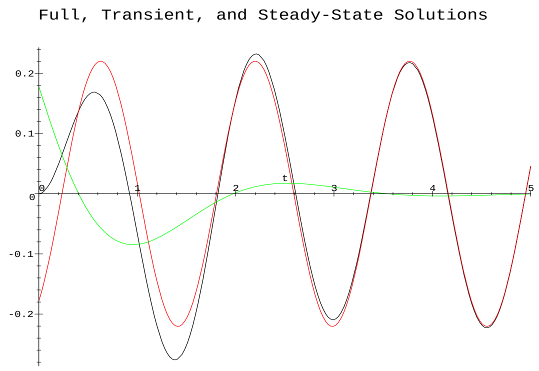
```
> gSOLN := 33/185*exp(-t)*cos(2*t)-63/370*exp(-t)*sin(2*t);
```

$$gSOLN := \frac{33}{185}\mathrm{e}^{(-t)}\cos(2t) - \frac{63}{370}\mathrm{e}^{(-t)}\sin(2t)$$

```
> pSOLN := -33/185*cos(4*t)+24/185*sin(4*t);
```

$$pSOLN := -\frac{33}{185}\cos(4t) + \frac{24}{185}\sin(4t)$$

```
> plot( { SOLN, gSOLN, pSOLN }, t=0..5,
>       title=`Full, Transient, and Steady-State Solutions` );
```



<u>Questions</u>

- what is the amplitude of the forcing term?
- what are the amplitude and phase angle of the steady-state solution?

```
> A[f] := 3;
```

$$A_f := 3$$

```
> A[ss] := sqrt( coeff( pSOLN, sin(4*t) )^2 + coeff( pSOLN, cos(4*t) )^2 );
> evalf( A[ss] );
```

$$A_{ss} := \frac{3}{185}\sqrt{185}$$

$$.2205643867$$

```
> phi[ss] := arctan( coeff( pSOLN, cos(4*t) ), coeff( pSOLN, sin(4*t) ) );
> evalf( phi[ss] );
```

$$\phi_{ss} := -\arctan\left(\frac{11}{8}\right)$$

$$-.9420000404$$

DEFINITION 1: The FREQUENCY GAIN (or GAIN FACTOR) is the ratio of the amplitudes of the steady-state solution and the forcing term.

In the current example, the frequency gain is

```
> M := A[ss]/A[f];
> evalf(M);
```

$$M := \frac{1}{185}\sqrt{185}$$

$$.07352146221$$

Note that this ratio is, in some sense, the fraction of the forcing term that that is present in the steady-state solution. In some situations it will be desired to maximize the frequency gain of a system.

The book's discussion of frequency gain points out that the gain factor depends, in part, on the frequency of the forcing term (and is independent of the amplitude of the forcing term). This suggests considering the same problem, without selecting a specific value for the frequency of the forcing function. That is,

```
> ODE2 := diff( x(t), t,t ) + 2*diff( x(t), t ) + 5*x(t) = 3*cos(gamma*t);
```

$$ODE2 := \left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + 2\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 5\,\mathrm{x}(t) = 3\cos(\gamma\, t)$$

(note that `gamma` refers to Euler's constant by default; the extra command at the beginning of the worksheet removes `gamma` from the list of system constants).

```
> EOM2 := dsolve( { ODE2, IC1, IC2 }, x(t) );
```

$$\begin{aligned}
EOM2 := \mathrm{x}(t) = {} & 3\,\frac{\cos(2\,t)\sin((\gamma+2)\,t)\,\gamma}{\%1} + \frac{9}{4}\,\frac{\cos(2\,t)\cos((\gamma-2)\,t)\,\gamma}{\%1} \\
& - \frac{3}{4}\,\frac{\cos(2\,t)\cos((\gamma-2)\,t)\,\gamma^3}{\%1} - \frac{3}{2}\,\frac{\cos(2\,t)\cos((\gamma-2)\,t)\,\gamma^2}{\%1} \\
& + \frac{3}{4}\,\frac{\cos(2\,t)\sin((\gamma-2)\,t)\,\gamma^2}{\%1} + \frac{15}{4}\,\frac{\sin(2\,t)\cos((\gamma+2)\,t)}{\%1} \\
& - \frac{9}{4}\,\frac{\cos(2\,t)\cos((\gamma+2)\,t)\,\gamma}{\%1} + \frac{15}{2}\,\frac{\sin(2\,t)\sin((\gamma+2)\,t)}{\%1} \\
& - \frac{15}{4}\,\frac{\cos(2\,t)\sin((\gamma+2)\,t)}{\%1} + \frac{15}{4}\,\frac{\cos(2\,t)\sin((\gamma-2)\,t)}{\%1} \\
& + \frac{15}{2}\,\frac{\cos(2\,t)\cos((\gamma+2)\,t)}{\%1} + \frac{15}{2}\,\frac{\cos(2\,t)\cos((\gamma-2)\,t)}{\%1} \\
& - \frac{9}{4}\,\frac{\sin(2\,t)\sin((\gamma+2)\,t)\,\gamma}{\%1} + \frac{3}{4}\,\frac{\sin(2\,t)\sin((\gamma+2)\,t)\,\gamma^3}{\%1} \\
& - \frac{3}{2}\,\frac{\sin(2\,t)\sin((\gamma+2)\,t)\,\gamma^2}{\%1} - \frac{15}{2}\,\frac{\sin(2\,t)\sin((\gamma-2)\,t)}{\%1} \\
& + 3\,\frac{\cos(2\,t)\sin((\gamma-2)\,t)\,\gamma}{\%1} + 3\,\frac{\sin(2\,t)\cos((\gamma-2)\,t)\,\gamma}{\%1} \\
& - \frac{9}{4}\,\frac{\sin(2\,t)\sin((\gamma-2)\,t)\,\gamma}{\%1} - 3\,\frac{\sin(2\,t)\cos((\gamma+2)\,t)\,\gamma}{\%1} \\
& + \frac{15}{4}\,\frac{\sin(2\,t)\cos((\gamma-2)\,t)}{\%1} + 3\,\frac{(\gamma^2-5)\;\mathrm{e}^{(-t)}\cos(2\,t)}{\%1} \\
& + \frac{3}{4}\,\frac{\cos(2\,t)\cos((\gamma+2)\,t)\,\gamma^3}{\%1} - \frac{3}{2}\,\frac{\cos(2\,t)\cos((\gamma+2)\,t)\,\gamma^2}{\%1} \\
& - \frac{3}{4}\,\frac{\cos(2\,t)\sin((\gamma+2)\,t)\,\gamma^2}{\%1} + \frac{3}{2}\,\frac{\sin(2\,t)\sin((\gamma-2)\,t)\,\gamma^2}{\%1} \\
& + \frac{3}{4}\,\frac{\sin(2\,t)\cos((\gamma+2)\,t)\,\gamma^2}{\%1} + \frac{3}{4}\,\frac{\sin(2\,t)\sin((\gamma-2)\,t)\,\gamma^3}{\%1}
\end{aligned}$$

$$+\frac{3}{4}\frac{\sin(2\,t)\cos(\,(\gamma-2\,)\,t\,)\,\gamma^2}{\%1}-\frac{3}{2}\frac{(\gamma^2+5)\,e^{(\,-t\,)}\sin(\,2\,t\,)}{\%1}$$
$$\%1 := 25-6\,\gamma^2+\gamma^4$$

This is even worse than before. Let's see if `convert(...+, trig)+` can transform this into a form we expect.

```
> SOLN2 := combine( rhs(EOM2), trig );
```

$$SOLN2 := -\frac{15}{2}\frac{e^{(\,-t\,)}\sin(\,2\,t\,)}{\%1}-15\frac{e^{(\,-t\,)}\cos(\,2\,t\,)}{\%1}-\frac{3}{2}\frac{e^{(\,-t\,)}\sin(\,2\,t\,)\,\gamma^2}{\%1}+3\frac{e^{(\,-t\,)}\cos(\,2\,t\,)\,\gamma^2}{\%1}$$
$$+15\frac{\cos(\,\gamma\,t\,)}{\%1}+6\frac{\gamma\sin(\,\gamma\,t\,)}{\%1}-3\frac{\gamma^2\cos(\,\gamma\,t\,)}{\%1}$$
$$\%1 := 25-6\,\gamma^2+\gamma^4$$

To be most effective, it would be nice to have all the terms involving the exponential solution and the different trigonometric terms grouped together. This can be done using the `collect` command.

```
> SOLN2 := collect( SOLN2,
>                   [ exp(-t), sin(2*t), cos(2*t),
>                     sin(gamma*t), cos(gamma*t) ] );
```

$$SOLN2 := \left(\left(-\frac{15}{2}\frac{1}{\%1}-\frac{3}{2}\frac{\gamma^2}{\%1}\right)\sin(\,2\,t\,)+\left(3\frac{\gamma^2}{\%1}-15\frac{1}{\%1}\right)\cos(\,2\,t\,)\right)e^{(\,-t\,)}$$
$$+6\frac{\gamma\sin(\,\gamma\,t\,)}{\%1}+\left(15\frac{1}{\%1}-3\frac{\gamma^2}{\%1}\right)\cos(\,\gamma\,t\,)$$
$$\%1 := 25-6\,\gamma^2+\gamma^4$$

As before, the transient and steady-state solutions are easily identified.

```
> gSOLN2 := ((-3/2/(25-6*gamma^2+gamma^4)*gamma^2-15/2/(25-6*gamma^2+gamma^4))
>              *sin(2*t)
>            +(-15/(25-6*gamma^2+gamma^4)+3/(25-6*gamma^2+gamma^4)*gamma^2)
>              *cos(2*t))*exp(-t);
```

$$gSOLN2 := \left(\left(-\frac{15}{2}\frac{1}{\%1}-\frac{3}{2}\frac{\gamma^2}{\%1}\right)\sin(\,2\,t\,)+\left(3\frac{\gamma^2}{\%1}-15\frac{1}{\%1}\right)\cos(\,2\,t\,)\right)e^{(\,-t\,)}$$
$$\%1 := 25-6\,\gamma^2+\gamma^4$$

```
> pSOLN2 := 6/(25-6*gamma^2+gamma^4)*gamma*sin(gamma*t)
>           +(-3/(25-6*gamma^2+gamma^4)*gamma^2+15/(25-6*gamma^2+gamma^4))
>             *cos(gamma*t);
```

$$pSOLN2 := 6\frac{\gamma\sin(\,\gamma\,t\,)}{25-6\,\gamma^2+\gamma^4}+\left(15\frac{1}{25-6\,\gamma^2+\gamma^4}-3\frac{\gamma^2}{25-6\,\gamma^2+\gamma^4}\right)\cos(\,\gamma\,t\,)$$

The amplitude of the steady-state solution is

```
> A[ss2] := sqrt( simplify( coeff( pSOLN2, sin(gamma*t) )^2
>                          + coeff( pSOLN2, cos( gamma*t) )^2 ) );
```

$$A_{ss2} := 3\sqrt{\frac{1}{25-6\,\gamma^2+\gamma^4}}$$

```
> phi[ss2] := arctan( coeff( pSOLN2, cos(gamma*t) ),
>                     coeff( pSOLN2, sin(gamma*t) ) );
```

$$\phi_{ss2} := \arctan\left(15\,\frac{1}{25-6\,\gamma^2+\gamma^4} - 3\,\frac{\gamma^2}{25-6\,\gamma^2+\gamma^4},\, 6\,\frac{\gamma}{25-6\,\gamma^2+\gamma^4}\right)$$

A quick check of these quantities when `gamma=4`, produces the same quantities as before

```
> subs( gamma=4, [ A[ss2], phi[ss2] ] );
```

$$\left[\frac{3}{185}\,\sqrt{185}, \arctan\left(\frac{-33}{185}, \frac{24}{185}\right)\right]$$

Moreover, the frequency gain for each value of γ is easily computed to be

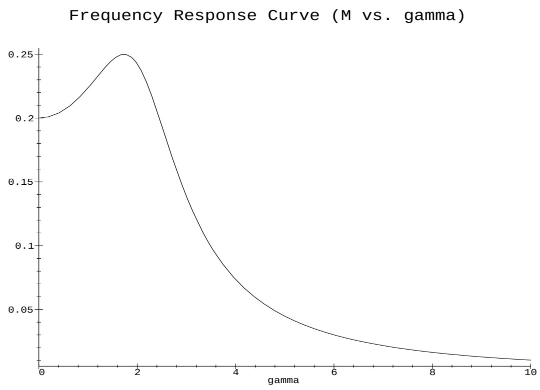
```
> M := A[ss2] / A[f] ;
```

$$M := \sqrt{\frac{1}{25-6\,\gamma^2+\gamma^4}}$$

Definition 2: The FREQUENCY RESPONSE CURVE or RESONANCE CURVE for a system is a graph of frequency gain versus the frequency of the forcing term.

In this example, the frequency response curve is

```
> plot( M, gamma=0..10,
>       title=`Frequency Response Curve (M vs. gamma)` );
```

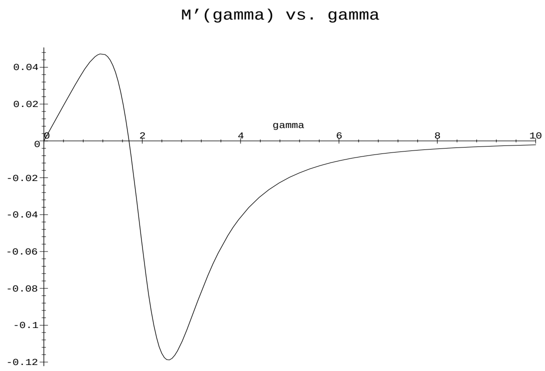


Note that the maximum frequency gain is ≈ 0.249 and occurs for $\gamma \approx 1.8$. The exact location of this maximizer can be found using Calculus. To begin, compute and plot the first derivative of the frequency gain.

```
> dM := diff( M, gamma );
```

$$dM := -\frac{1}{2}\,\frac{-12\,\gamma+4\,\gamma^3}{\sqrt{\frac{1}{25-6\,\gamma^2+\gamma^4}}\,(25-6\,\gamma^2+\gamma^4)^2}$$

```
> plot( dM, gamma=0..10,
>       title=`M'(gamma) vs. gamma` );
```



The maximum frequency gain occurs for a (positive) frequency where the derivative has a zero. These points are found to be

```
> solve( dM=0, { gamma } );
```

$$\{\gamma = 0\}, \{\gamma = \mathrm{RootOf}(-3 + \_Z^2)\}$$

```
> allvalues( "[2] );
```

$$\left\{\gamma = \sqrt{3}\right\}, \left\{\gamma = -\sqrt{3}\right\}$$

Thus, the maximum frequency gain occurs when

```
> gamma_r := "[1];
```

$$gamma\_r := \left\{\gamma = \sqrt{3}\right\}$$

and the maximum frequency gain is

```
> simplify( subs( gamma_r, M ) );
```

$$\frac{1}{4}$$

or 25% of the forcing function's amplitude.

Questions:

- what is the phase angle when the frequency gain is maximized?
- how does the phase angle vary with the frequency, γ?
- describe the small and large frequency behavior of the phase angle

DEFINITION 3: The RESONANCE FREQUENCY of a system is the frequency, $\frac{\gamma_r}{2\pi}$, at which the maximum frequency response is obtained.

For the example considered here, resonance occurs when

```
> RESONANCE := subs( gamma_r, gamma/(2*Pi) );
```

$$RESONANCE := \frac{1}{2} \frac{\sqrt{3}}{\pi}$$

The discussion in the text concludes with a brief investigation of the dependence of the resonance frequency on the damping coefficient. This is omitted from the worksheet, but you are still responsible for understanding this relationship.