Discontinuous Coefficients and Forcing Functions (§§2.3, 3.4)

DISCONTINUOUS COEFFICIENTS AND FORCING FUNCTIONS (§2.3, #31 and 32, pp. 53–54)

```
> restart;
> with( plots ):
> with( DEtools ):
```

Differential equations with piecewise-defined coefficients (including the forcing function) arise in a number of natural circumstances. For example, the air resistance encountered by a skydiver is (dramatically) different between the free-fall and full deployment stages of the jump (see Example 3, pp. 112–113)..

We will use graphical solution methods to first see that the solution to such a problem is the same as before: solutions must follow the direction field. In particular, even though the direction field may not be continuous, the solution may still be continuous (as would be expected in the skydiving example). The issue of finding an explicit formula for the solution is a little different. The key is to break the problem into pieces where the solution can be determined, then to piece these solutions together with appropriate transition conditions.

The discussion follows the general steps found in Problems 31 and 32 from Section 2.3 Nagle and Saff.

Objectives:

- understand how ODEs with discontinuous coefficients might arise in a physical application
- develop conceptual and mechanical skills for the analysis of ODEs with piecewise-defined coefficients
- continue to develop Maple skills for the manipulation of differential equations
- continue to built a geometric intuition about ODEs and IVPs

New Maple Commands:

`piecewise` – creates a piecewise-defined function, *i.e.*, a function defined by different rules for different parts

Problem 31 (Section 2.3, p. 53) — Discontinuous Coefficients

There are times when the coefficient $P(x)$ in a linear equation may not be continuous but may have a jump discontinuity. Fortunately, we may still obtain a "reasonable" solution. For example, consider the initial value problem:

$$\frac{dy}{dx} + P(x)y = x, \qquad\qquad y(0) = 1$$

where

$$P(x) = \begin{cases} 1, & 0 \le x \le 2 \\ 3, & x > 2 \end{cases}$$

Prior to following the book's outline for the computation of an explicit solution, let's spend a little time considering exactly what the solution should look like. For $0 \le x \le 2$, the problem is simply a standard IVP for a first-order linear equation with constant coefficients. For $x > 2$ a different ODE is involved, but it is still a first-order linear equation with constant coefficients. The general solution to either problem is simple to determine. The question is, exactly which of these solutions should be selected to form the solution to the IVP?

The initial condition $y(0) = 1$ identifies the specific solution curve for $0 \le x \le 2$. The solution curve followed for $x > 2$ is the one that meets the solution at the end of the first stage. That is, the solution from the first stage is used to determine the initial condition for the second stage. (The precise definition of this will be clearer later.)

But first, let's develop some intuition by considering the direction field for this equation. First, note that the coefficient of $y(x)$ is just P, not $P(x)$, and is defined using the verb+piecewise+ command:

```
> P := piecewise( x<=2, 1, 3 );
```

Then, the differential equation and initial condition can be represented as follows.

```
> ODE := diff( y(x), x ) + P*y(x) = x;
```

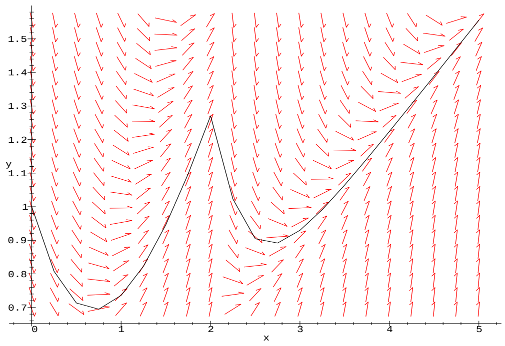
$$ODE := \left(\frac{\partial}{\partial x}\,\mathrm{y}(\,x\,)\right) + \mathrm{piecewise}(\,0, x \le 2, 1, 3\,)\,\mathrm{y}(\,x\,) = x$$

```
> IC := y(0)=1;
```

$$IC := \mathrm{y}(\,0\,) = 1$$

The direction field for this problem is then created exactly as all previous direction fields have been created

```
> DEplot( ODE, [x,y], x=0..5, { [0,1] }, arrows=THIN,
>         title=`Problem 31 (Section 2.3) -- Direction Field and Solution` );
```



Note that the discontinuity in the direction field at $x = 2$ is quite noticeable. Note also how the solution, which follows the direction field at all points, is continuous. The remainder of this problem is concerned with finding an explicit solution to this problem.

(a) Find the general solution for $0 \le x \le 2$.

For these values of x, $P(x) = 1$ and the integrating factor is $\mu(x) = e^x$. The general solution is:

```
> ODE1 := diff( y(x), x ) + y(x) = x;
```

$$ODE1 := \left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + \mathrm{y}(x) = x$$

```
> GSOLN1 := dsolve( ODE1, y(x) );
```

$$GSOLN1 := \mathrm{y}(x) = x - 1 + \mathrm{e}^{(-x)}\,\_C1$$

(b) Choose the constant in the solution of part (a) so that the initial condition is satisfied.

The initial condition is $y(0) = 1$. The appropriate value of `_C1` can be found as follows

```
> CEQN1 := subs( x=0, IC, GSOLN1 );
```

$$CEQN1 := 1 = -1 + \mathrm{e}^0\,\_C1$$

```
> CSOLN1 := solve( CEQN1, { _C1 } );
```

$$CSOLN1 := \{\_C1 = 2\}$$

```
> SOLN1 := subs( CSOLN1, GSOLN1 );
```

$$SOLN1 := \mathrm{y}(x) = x - 1 + 2\,\mathrm{e}^{(-x)}$$

or, `dsolve` could be used to find the solution in one clean step

```
> dsolve( { ODE1, IC }, y(x) );
```

$$\mathrm{y}(x) = x - 1 + 2\,\mathrm{e}^{(-x)}$$

(c) Find the general solution for $x > 2$.

The differential equation for these values of x are

```
> ODE2 := diff( y(x), x ) + 3*y(x) = x;
```

$$ODE2 := \left(\frac{\partial}{\partial x}\,\mathrm{y}(\,x\,)\right) + 3\,\mathrm{y}(\,x\,) = x$$

```
> GSOLN2 := dsolve( ODE2, y(x) );
```

$$GSOLN2 := \mathrm{y}(\,x\,) = \frac{1}{3}\,x - \frac{1}{9} + \mathrm{e}^{(\,-3\,x\,)}\,\_C1$$

(d) Now choose the constant in the general solution from part (c) so that the solution from part (b) and the solution from part (c) agree at $x = 2$. By patching the two solutions together, we are able to obtain a continuous function that satisfies the differential equation except at $x = 2$, where the derivative is undefined.

This is telling us to use the value of the first solution at $x = 2$ as the initial condition for the second equation. To implement this idea we need to know this "initial condition". This is found quite easily:

```
> IC2 := subs( x=2, SOLN1 );
```

$$IC2 := \mathrm{y}(\,2\,) = 1 + 2\,\mathrm{e}^{(\,-2\,)}$$

This initial condition can then be used to find the solution which agrees with the solution during the first stage

```
> CEQN2 := subs( x=2, IC2, GSOLN2 );
```

$$CEQN2 := 1 + 2\,\mathrm{e}^{(\,-2\,)} = \frac{5}{9} + \mathrm{e}^{(\,-6\,)}\,\_C1$$

```
> CSOLN2 := solve( CEQN2, { _C1 } );
```

$$CSOLN2 := \left\{\_C1 = \frac{\frac{4}{9} + 2\,\mathrm{e}^{(\,-2\,)}}{\mathrm{e}^{(\,-6\,)}}\right\}$$

```
> SOLN2 := simplify( subs( CSOLN2, GSOLN2 ) );
```

$$SOLN2 := \mathrm{y}(\,x\,) = \frac{1}{3}\,x - \frac{1}{9} + \frac{4}{9}\,\mathrm{e}^{(\,-3\,x+6\,)} + 2\,\mathrm{e}^{(\,-3\,x+4\,)}$$

Or, more directly,

```
> dsolve( { ODE2, IC2 }, y(x) );
```

$$\mathrm{y}(\,x\,) = \frac{1}{3}\,x - \frac{1}{9} - \frac{\mathrm{e}^{(\,-3\,x\,)}\left(-\frac{4}{9} - 2\,\mathrm{e}^{(\,-2\,)}\right)}{\mathrm{e}^{(\,-6\,)}}$$

```
> simplify(" );
```

$$\mathrm{y}(\,x\,) = \frac{1}{3}\,x - \frac{1}{9} + \frac{4}{9}\,\mathrm{e}^{(\,-3\,x+6\,)} + 2\,\mathrm{e}^{(\,-3\,x+4\,)}$$

Thus, the continuous solution to this problem is the solution formed by piecing together the solutions for $0 \le x \le 2$ and for $x > 2$. This is most conveniently accomplished using the `piecewise` command:

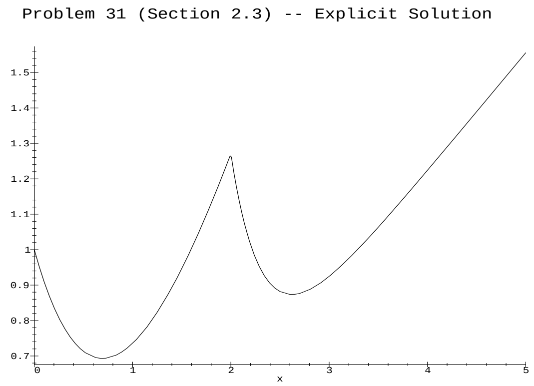
```
> SOLN := piecewise( x<=2, rhs(SOLN1), rhs(SOLN2) );
```

$$SOLN := \text{piecewise}\left(0, x \leq 2, x - 1 + 2\,e^{(-x)}, \frac{1}{3}\,x - \frac{1}{9} + \frac{4}{9}\,e^{(-3\,x+6)} + 2\,e^{(-3\,x+4)}\right)$$

(e) Sketch the graph of the solution from $x = 0$ to $x = 5$.

There are a number of ways in which the graph can be constructed. One of the most efficient, and straightforward, is to use the previously defined piecewise-defined solution, SOLN. Then the requested plot is obtained with a single `plot` command

```
> plot( SOLN, x=0..5,
>       title=`Problem 31 (Section 2.3) -- Explicit Solution` );
```



Note that this solution agrees with the graphical solution obtained at the beginning of the problem.

Problem 32 (Section 2.3, p. 54) — Discontinuous Forcing Terms

There are times when the forcing term $Q(x)$ in a linear equation may not be continuous but may have a jump discontinuity. Fortunately, we may still obtain a reasonable solution imitating the procedure discussed in Problem 31. Use this procedure to find the continuous solution to the initial value problem

$$\frac{dy}{dx} + 2y = Q(x), \qquad y(0) = 0$$

where

$$Q(x) = \begin{cases} 2, & 0 \leq x \leq 3 \\ -2, & x > 3 \end{cases}$$

That is,

```
> Q := piecewise( x<=3, 2, -2 );
```

$$Q := \text{piecewise}(\,0, x \leq 3, 2, -2\,)$$

Sketch the graph of the solution from $x = 0$ to $x = 7$.

The Maple implementation of this problem is

```
> ODE := diff( y(x), x ) + 2*y(x) = Q;
```

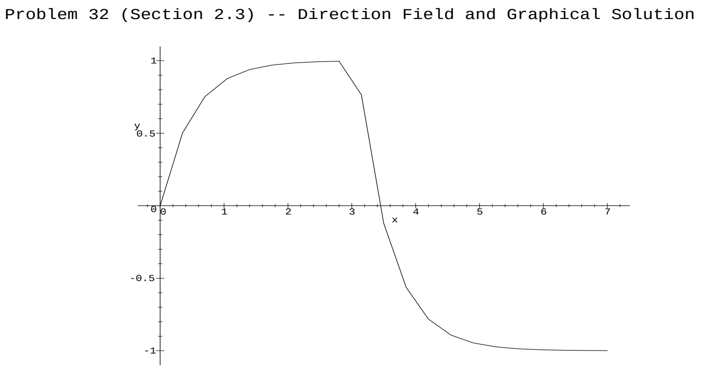
$$ODE := \left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 2\,\mathrm{y}(x) = \mathrm{piecewise}(0, x \le 3, 2, -2)$$

```
> IC := y(0) = 0;
```

$$IC := \mathrm{y}(0) = 0$$

The direction field and graphical solution to this IVP are produced by one call to `DEplot`:

```
> DEplot( ODE, [x,y], x=0..7, { [0,0] },
>         title=`Problem 32 (Section 2.3) -- `
>               .`Direction Field and Graphical Solution` );
```



Note the low resolution of this graph. (This could be improved by specifying the `stepsize=` optional argument.) Regardless, it seems as though the solution to the first solution wants to be asymptotic to $y = 1$ initially, but the jump in Q pushes the solutions towards $y = -1$ for large values of t. The explicit solution should confirm these observations.

The explicit solution is found by following the outline introduced in Problem 31. Both the "long" and the "short" methods will be demonstrated. The "long approach" begins by finding the general solution on $0 \le x \le 3$:

```
> ODE1 := diff(  y(x), x ) + 2*y(x) = 2;
```

$$ODE1 := \left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 2\,\mathrm{y}(x) = 2$$

Then, find the value of the constant, `_C1`, that satisfies the initial condition

```
> GSOLN1 := dsolve( ODE1, y(x) );
```

$$GSOLN1 := \mathrm{y}(x) = 1 + \mathrm{e}^{(-2\,x)}\,\_C1$$

```
> CEQN1 := subs( x=0, IC, GSOLN1 );
```

$$CEQN1 := 0 = 1 + \mathrm{e}^{0}\,\_C1$$

```
> CSOLN1 := solve( CEQN1, { _C1 } );
```

$$CSOLN1 := \{\_C1 = -1\}$$

```
> SOLN1 := subs( CSOLN1, GSOLN1 );
```

$$SOLN1 := \mathrm{y}(x) = 1 - \mathrm{e}^{(-2x)}$$

The general solution to the differential equation for $x > 3$ is found as follows:

```
> ODE2 := diff( y(x), x ) + 2*y(x) = -2;
```

$$ODE2 := \left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 2\,\mathrm{y}(x) = -2$$

```
> GSOLN2 := dsolve( ODE2, y(x) );
```

$$GSOLN2 := \mathrm{y}(x) = -1 + \mathrm{e}^{(-2x)}\,\_C1$$

To ensure continuity of the final solution, the constant `_C1` must be chosen so the solution for $x > 3$ agrees with the solution for $0 \le x \le 3$ at the boundary point $x = 3$. Thus,

```
> IC2 := subs( x=3, SOLN1 );
```

$$IC2 := \mathrm{y}(3) = 1 - \mathrm{e}^{(-6)}$$

```
> CEQN2 := subs( x=3, IC2, GSOLN2 );
```

$$CEQN2 := 1 - \mathrm{e}^{(-6)} = -1 + \mathrm{e}^{(-6)}\,\_C1$$

```
> CSOLN2 := solve( CEQN2, { _C1 } );
```

$$CSOLN2 := \left\{\_C1 = \frac{2 - \mathrm{e}^{(-6)}}{\mathrm{e}^{(-6)}}\right\}$$

```
> SOLN2 := simplify( subs( CSOLN2, GSOLN2 ) );
```

$$SOLN2 := \mathrm{y}(x) = -1 + 2\,\mathrm{e}^{(-2x+6)} - \mathrm{e}^{(-2x)}$$

The "short" approach is characterized by working with the solution to an initial value problems, rather than the general solutions to an ODE. The first step is to solve the initial value problem for $0 \le x \le 3$:

```
> SOLN1a := dsolve( { ODE1, IC }, y(x) );
```

$$SOLN1a := \mathrm{y}(x) = 1 - \mathrm{e}^{(-2x)}$$

The initial condition for the second stage is found as before:

```
> IC2a := subs( x=3, SOLN1a );
```

$$IC2a := \mathrm{y}(3) = 1 - \mathrm{e}^{(-6)}$$

```
> SOLN2a := dsolve( { ODE2, IC2 }, y(x) );
```

$$SOLN2a := \mathrm{y}(x) = -1 - \frac{\mathrm{e}^{(-2x)}\,(-2 + \mathrm{e}^{(-6)})}{\mathrm{e}^{(-6)}}$$

```
> simplify( " );
```

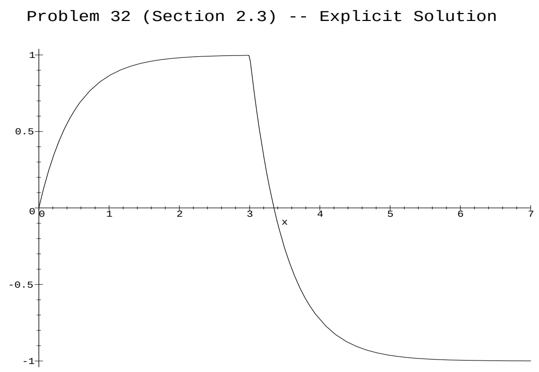
$$\mathrm{y}(x) = -1 + 2\,\mathrm{e}^{(-2x+6)} - \mathrm{e}^{(-2x)}$$

In either case, the complete solution can be expressed using `piecewise`:

```
> SOLN := piecewise( x<=3, rhs(SOLN1), rhs(SOLN2) );
```

$$SOLN := \text{piecewise}(\,0, x \leq 3, 1 - e^{(\,-2\,x\,)}, -1 + 2\,e^{(\,-2\,x+6\,)} - e^{(\,-2\,x\,)}\,)$$

```
> plot( SOLN, x=0..7,
>       title=`Problem 32 (Section 2.3) -- Explicit Solution` );
```



Note that this plot has better resolution, and exhibits the asymptotic behaviors identified earlier.