Basic MAPLE Demonstration for Students

(A BRIEF) INTRODUCTION TO MAPLE: Basic Manipulations and Plots

What is Computer Algebra?

Much of your mathematical background has probably be focused on developing the ability to solve equations and analyze functions. The sophisticated manipulation of symbols and expressions that you use to solve equations and investigate functions can also be performed by a computer algebra system (CAS). A CAS can be use to generate many of the exact symbolic solutions typically obtained by hand, the numerical approximations found using a calculator, and the graphs sketched by hand. A CAS is a software package that works directly with an expression as a basic data structure (compare this to most software packages that require numeric data and perform numeric operations on the data).

Maple is one of several CASs; other major examples are Mathematica, Macsyma, Derive, and MathCad. The ability to work directly with mathematical expressions means that floating point errors arise only when floating point operations are explicitly requested.

Documentation and On-Line Help

The on-line help for Maple is very good – once you learn how to use it. The help pages describe the syntax of each command, a brief description of the algorithm used to implement the command, and a few examples illustrating the use of the command. The examples can be very helpful. I will often cut and paste an example from a help page into my worksheet, confirm that the example works as advertised, then modify the arguments to answer my question.

To obtain help on the command `func`, type `?func` or `help func` at a Maple prompt (which, typically, is `>`).

Of course, this works only if the name of the function, or a reasonable keyword associated with the name, is known. The Keyword Search (located under the Help menu on the title bar) presents a hierarchical view of the help documents sorted by general topic. The Quick Reference pages provided together with this document are also quite helpful.

It is a good idea to begin EVERY worksheet with the command:

```
> restart:
```

This ensures that all of Maple's memory is cleared. Future operations can be performed without risk that previous definitions will interfere with the processing.

Essential Facts

Maple is a computer language; it cannot read your mind. Maple does follow your instructions to the letter. This means that you need to learn to communicate your needs to Maple in a way that it understands.

Some basic symbols and other fundamentals:

- assignments are made with `:=` (plain `=` has a different meaning) – this gives the value of the right hand side (RHS) to the name on the LHS
- every command is terminated by a semi-colon (`;`) or colon (`:`) – with the latter the computation is done, but the result is not displayed
- the double quote (") refers to the result of the immediately preceding computation
- Maple is case sensitive – that is, the names `x` and `X` are different, `pi` and `Pi` are not the same thing
- `{ }` – set notation (members of a set are unordered)
- `..` (as in `a .. b`) – this is how Maple indicates the interval $[a, b]$, that is, the real numbers from a to b
- `?` or `help` – a request to Maple for information

The standard mathematical functions are denoted by their standard names: `+` (plus) , `-` (minus) , `*` (times) , `/` (divided by) , `^` (raised to the power) , `sin`, `cos`, `tan`, `abs`, `sqrt`

Common constants are predefined in Maple, their names are: `Pi`, `E`, `I`, and `infinity` (Yes, they have to be capitalized exactly like this.)

Unless specified otherwise (with `assume`), all names are assumed to be complex valued. This is not often a problem, but there are times when this can lead to surprising results. (We'll see an example of this during this demonstration.)

Arithmetic

In the following lines, use paper and pencil to first predict what you think Maple will do. Then execute the command and see what actually happens!

```
> number := 4 *  6 + 12 /  6 - 1 ;
> power := (-3)^3 ;
> abs( " );
> Pi ;
> v:= sin( Pi / 4 ) :
> w:= v^2  ;
> v;
> tan(  -Pi / 2 ) ;
> 3 / ( 5 - sqrt( number ) ) ;
```

Now enter some commands of your own! To produce the input prompt either select Insert New Region (Below) from the Format menu (or, in `xmaple`, Insert Prompt from the Edit menu), click on the eighth icon from the left, or simply press ALT-I. Any of these will produce a new input region below the current cursor position. Be generous with the

space bar to make your commands easy to read, and to edit (modify after the fact). Go back to the tangent calculation above and change it to compute the tangent of Pi / 3.

Now for some slightly more impressive computations

```
> 2^100;
> evalf( Pi, 250 );
> 400!;
```

Now let's define a few functions and see how Maple operates with these. (The only difference is that in these examples the name x does not have a value.)

```
> f := sin( 2  * x ) ;
```

$$f := \sin(2\,x)$$

```
> g := 4 * x^2 ;
```

$$g := 4\,x^2$$

```
> h := 0.25 * cos ( 8 * x ) ;
```

$$h := .25\cos(8\,x)$$

```
> p := f * g ;
```

$$p := 4\sin(2\,x)\,x^2$$

```
> u1 :=  f * h ;
```

$$u1 := .25\sin(2\,x)\cos(8\,x)$$

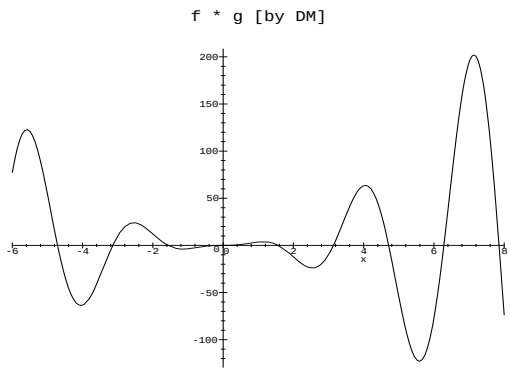
```
> u2 := f + h ;
```

$$u2 := \sin(2\,x) + .25\cos(8\,x)$$

Plotting

Graphs of functions are produced by the plot command. In its simplest form, plot needs to know the function to be plotted and the range of values for the independent variable. Note that a..b is Maple's way of describing the interval [a,b]. Observe that I have placed signed titles on some of these graphs. When you actually print plots to turn in with an assignment, they MUST have a title and signature [by your initials or last name]. Again, try to predict the output before tapping the return key!

```
> plot( 3 * t - 2 , t = -3 .. 10, title=‘My First Maple Plot‘ ) ;
> plot( f , x = -Pi .. 2 * Pi ) ;
> plot( p , x = -6 .. 8 , title = ‘ f * g [by DM] ‘ );
```

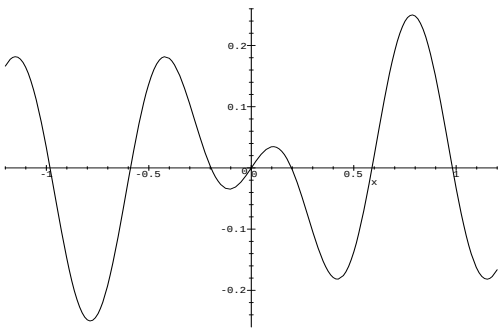


These plots are nice, but what information do they convey? Let's concentrate on the plot of f*g, that is, of p. Position the cursor on a point on the graph and click the left button. The numbers that appear on the lower-left corner (upper-left, in xmaple) are the coordinates of the current location of the cursor. Use this technique to identify the global maximum and minimum values of f*g on the interval [-6, 8], and the x-values at which these are found. Where do other (local) minima and maxima occur? Can you guess the exact values?

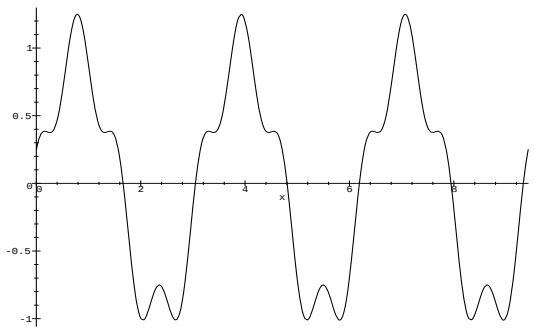
```
> plot( 2 *  u^3 + 4 * u - 5 , u = -10  .. 7 ,
>       title = ‘ cubic function [by DM] ‘ ) ;
```

Where does the cubic function cross the u-axis? Maybe it would be helpful to cut down the plotting interval from [-10, 7] to [-1, 3], or even narrower, say to [0, 1.5]. Try it. Can you find the u-intercept to 2-decimal point accuracy by this process? It is a powerful method, which we call ZOOMING IN.

```
> plot( u1 ,  x = -1.2 .. 1.2 );
```




```
> plot( u2 , x = 0 .. 3*Pi ) ;
```

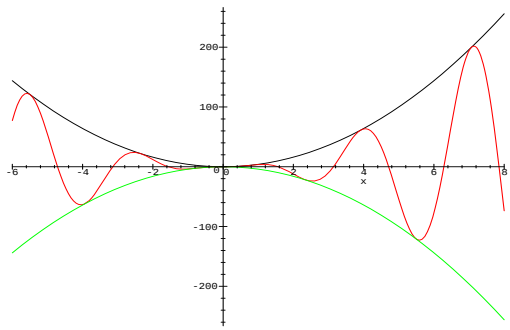


Can you explain why the graphs of u1 and u2 look the way they do?

You probably now have a big pile of plots cluttering up your screen. Ones that you don't need can be closed temporarily by minimizing the window. To permanently get rid of plots that you no longer need, position the cursor inside the plot window, select Exit from the File menu on the plot window or use the appropriate key sequence.

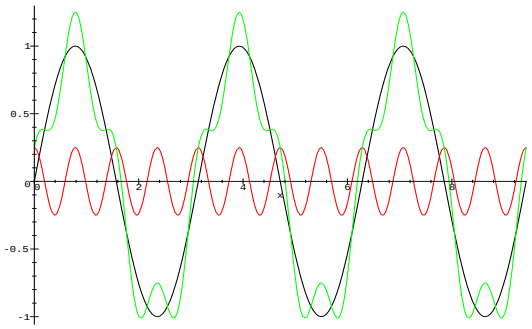
To plot more than one function in a single window, just specify the functions to be plotted as a set.

```
> plot( { p , g , -g } ,  x= -6 ..  8 );
```



Can you identify the three different plots? What about the next one? Can you explain the little bumps that appear inside the large dips in the graph of s?

```
> plot( { f , h , u2 }, x=0 .. 3*Pi );
```



Discontinuous Functions

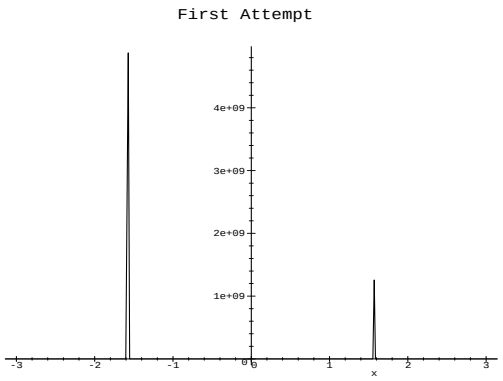
The tangent function is well-known to all of us. Note: This definition completely replaces the previous definition of f.

```
> f :=  tan(x);
```

$$f := \tan(x)$$

Recall that the graph of y=tan(x) has vertical asymptotes at all odd multiples of Pi/2. Let's see how Maple handles this.

```
> plot( f , x = -Pi .. Pi, title=`First Attempt` );
```



Is this what you expected to see? Why does the graph look like this? What can be done to improve the appearance of the plot?

There are two problems. First, we have to tell Maple that this function is discontinuous. But, there is more. Look at the labels on the vertical axis. The units are (on most platforms) in BILLIONS; we want to see the details on a much finer scale. Let's limit the vertical scale to the range from -10 to 10. Each of these pieces of information is an optional argument to plot.

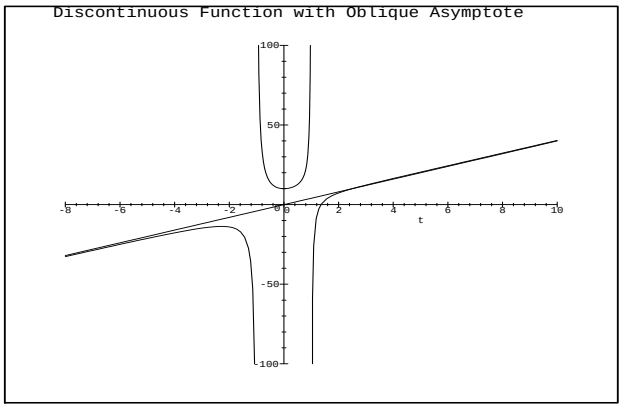
```
> plot( f,  x = -Pi .. Pi ,  -10 .. 10 , discont=true );
```

This technique is very useful for any function that has vertical asymptotes.

OK. Now it's your turn. Add appropriate optional arguments to the following command to produce a graph from which you can identify the local minimum and local maximum. (Don't forget the title.)

```
> plot ( ( 10 -  4*t^3) / ( 1 - t^2 ) , t = -3 .. 5 );
```

Now, one last challenge. Use your graph to estimate the (oblique) asymptote for this function. Check your estimate by graphing the function and the asymptote in the same plot. (Here's an example of what I'm after, and to show how plots look when inserted into a worksheet.)



These examples only scratch the surface of Maple's abilities when it comes to plotting. Maple can also produce 3-D plots (see `?plot3d`), plots of parametric curves and surfaces, animated sequences (see `?animate` or `?display` or `?animate3d` or `?display3d`). The best general source of information is the on-line help for the `plots` package.

```
> ?plots
```

Calculus Manipulations

Consider the following sequence of operations relating to the function

$$x(t) = e^{-\alpha t}\cos(\omega * t + \theta).$$

What information is obtained in each step?

```
> restart;
> x := exp(-alpha*t)*cos(omega*t+theta);
```

$$x := \mathrm{e}^{(-\alpha\, t)}\cos(\omega\, t + \theta)$$

```
> dx := diff( x, t );
```

$$dx := -\alpha\,\mathrm{e}^{(-\alpha\, t)}\cos(\omega\, t + \theta) - \mathrm{e}^{(-\alpha\, t)}\sin(\omega\, t + \theta)\,\omega$$

```
> dx2 := diff( x, t$2 );
```

$$dx2 := \alpha^2 \, e^{(-\alpha\, t)} \cos(\omega\, t + \theta) + 2\,\alpha\, e^{(-\alpha\, t)} \sin(\omega\, t + \theta)\,\omega - e^{(-\alpha\, t)} \cos(\omega\, t + \theta)\,\omega^2$$

```
> factor( dx );
```

$$-e^{(-\alpha\, t)} \left(\alpha \cos(\omega\, t + \theta) + \sin(\omega\, t + \theta)\,\omega\right)$$

```
> CP := solve( dx=0, t );
```

$$CP := -\frac{\theta + \arctan\left(\frac{\alpha}{\omega}\right)}{\omega}$$

```
> subs( t=CP, dx2 );
```

$$\alpha^2 \, e^{\left(\frac{\alpha\,(\theta + \arctan(\frac{\alpha}{\omega}))}{\omega}\right)} \cos\left(-\arctan\left(\frac{\alpha}{\omega}\right)\right) + 2\,\alpha\, e^{\left(\frac{\alpha\,(\theta + \arctan(\frac{\alpha}{\omega}))}{\omega}\right)} \sin\left(-\arctan\left(\frac{\alpha}{\omega}\right)\right)\omega$$
$$- e^{\left(\frac{\alpha\,(\theta + \arctan(\frac{\alpha}{\omega}))}{\omega}\right)} \cos\left(-\arctan\left(\frac{\alpha}{\omega}\right)\right)\omega^2$$

```
> dx := simplify( ", trig );
```

$$dx := -\frac{e^{\left(\frac{\alpha\,(\theta + \arctan(\frac{\alpha}{\omega}))}{\omega}\right)} \left(\omega^2 + \alpha^2\right)}{\sqrt{\frac{\omega^2 + \alpha^2}{\omega^2}}}$$

Since the derivative will be negative (for real values of omega, alpha, and theta), this critical point must be a (local) maximum. The value of the function at this maximum is

```
> X[max] := simplify( subs( t=CP, x ), trig );
```

Can the same information be reached using the Second Derivative Test? (What is the second derivative of `x`?)

Indefinite and definite integration is obtained in a natural way.

```
> Int( x, t );
```

$$\int e^{(-\alpha\, t)} \cos(\omega\, t + \theta)\, dt$$

```
> value( "  );
```

$$-\frac{\alpha\, e^{(-\alpha\, t)} \cos(\omega\, t + \theta)}{\omega^2 + \alpha^2} + \frac{\omega\, e^{(-\alpha\, t)} \sin(\omega\, t + \theta)}{\omega^2 + \alpha^2}$$

```
> simplify( " );
```

$$-\frac{e^{(-\alpha\, t)} \left(\alpha \cos(\omega\, t + \theta) - \sin(\omega\, t + \theta)\,\omega\right)}{\omega^2 + \alpha^2}$$

Note that the constant of integration is not included in these results.

Definite integrals are very similar. Here's an example taken from the CRC Tables of Integrals.

```
> Int( u/sin(u), u=0..Pi/2 ):
> " = value( " );
```

$$\int_0^{1/2\,\pi} \frac{u}{\sin(u)}\,du = \frac{1}{8}\,I\,\pi^2 + 2\,Catalan + \frac{1}{4}\,\pi\ln(2) - \frac{1}{2}\,\pi\ln(1+I)$$

The presence of complex numbers in this result is (at first) surprising. It is not difficult to see that this quantity is real. (Note the use of `:` and `=` to improve the appearance of the output.)

```
> evalc( " );
```

$$\int_0^{1/2\,\pi} \frac{u}{\sin(u)}\,du = 2\,Catalan$$

Now, what is `Catalan`? (see `?Catalan`) A numerical (approximation to the) value of this integral is

```
> evalf( rhs(") );
```

$$1.831931188$$

This example illustrates a very important feature of Maple. Even though a problem is stated in terms of real quantities, the computational algorithms sometimes require the use of complex analysis. It is also very important to remember that all names are, by default, assumed to be complex valued. Many common calculations actually require an additional assumption on a parameter. This can be very frustrating. (It is one of the most common questions that I see on the Maple User's Group.)

To illustrate, consider the following two improper integrals:

```
> int( exp(-u^2), u=0..infinity );
```

$$\frac{1}{2}\,\sqrt{\pi}$$

```
> int( exp(-a*u^2), u=0..infinity );
```

$$\lim_{u\to\infty-} \frac{1}{2}\,\frac{\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{a}\,u\right)}{\sqrt{a}}$$

Note that Maple is unable to evaluate this limit until something is known about the parameter a.

```
> assume( a>0 );
> about( a );
Originally a, renamed a~:
  is assumed to be: RealRange(Open(0),infinity)


> int( exp(-a*u^2), u=0..infinity );
```

$$\frac{1}{2}\,\frac{\sqrt{\pi}}{\sqrt{\tilde{a}}}$$

The following assignment is used to reset the name `a` to its unassigned state.

```
> a := 'a';
> about( a );
```

$$a := a$$

```
a:
  nothing known about this object
```

Here is a simple limit example.

```
> L := Limit( tan(theta), theta=Pi/2 );
```

$$L := \lim_{\theta \to (1/2\,\pi)} \tan(\theta)$$

```
> value( L );
```

$$undefined$$

```
> limit( tan(theta), theta=Pi/2, right );
```

$$-\infty$$

Application I: Integration of a Rational Function

```
> restart;
```

Consider the proper rational function

```
> f := (x^2-26*x-47)/(x^5+5*x^3+3*x^4+11*x^2-20);
```

$$f := \frac{x^2 - 26\,x - 47}{x^5 + 5\,x^3 + 3\,x^4 + 11\,x^2 - 20}$$

The antiderivative of `f` can be obtained by a variety of different means. Let's examine a few and compare the information obtained in each case.

```
> F := Int( f, x );
```

$$F := \int \frac{x^2 - 26\,x - 47}{x^5 + 5\,x^3 + 3\,x^4 + 11\,x^2 - 20}\,dx$$

The simplest, but least instructive, approach is to simply let Maple do all the work.

Note that the constant of integration is not included in indefinite integrals.

```
> F  = value( F );
```

$$\int \frac{x^2 - 26\,x - 47}{x^5 + 5\,x^3 + 3\,x^4 + 11\,x^2 - 20}\,dx =$$
$$-\frac{4}{3}\ln(x-1) + \frac{1}{3}\,\frac{1}{x+2} + \frac{23}{27}\ln(x+2) + \frac{13}{54}\ln(x^2+5) + \frac{91}{135}\sqrt{5}\arctan\left(\frac{1}{5}\,x\,\sqrt{5}\right)$$

or, even,

```
> int( f, x );
```

$$-\frac{4}{3}\ln(x-1) + \frac{1}{3}\,\frac{1}{x+2} + \frac{23}{27}\ln(x+2) + \frac{13}{54}\ln(x^2+5) + \frac{91}{135}\sqrt{5}\arctan\left(\frac{1}{5}\,x\,\sqrt{5}\right)$$

To emphasize integration techniques while still avoiding algebraic complications, we can utilize Maple to determine the partial fraction expansion of the integrand.

```
> fpf := convert( f, parfrac, x );
```

$$fpf := -\frac{4}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{1}{(x+2)^2} + \frac{23}{27}\,\frac{1}{x+2} + \frac{13}{27}\,\frac{7+x}{x^2+5}$$

From this partial fraction decomposition it is now a relatively simple matter to compute

the integral BY HAND. The result can be compared with the previous answer

```
> int( fpf, x );
```

$$-\frac{4}{3}\ln(x-1)+\frac{1}{3}\frac{1}{x+2}+\frac{23}{27}\ln(x+2)+\frac{13}{54}\ln(x^2+5)+\frac{91}{135}\sqrt{5}\arctan\left(\frac{1}{5}x\sqrt{5}\right)$$

The third, and final, approach to this problem will be the most detailed. The purpose here is to emphasize the process of finding a partial fraction expansion without fear of unmanageable algebra.

The key here is to enter the correct form for the partial fraction decomposition. This depends upon the irreducible factors in the denominator of `f`:

```
> factor( denom( f ) );
```

$$(x-1)(x^2+5)(x+2)^2$$

The appropriate general form for the decomposition will be

```
> FORM := a/(x-1) + b/(x+2) + c/(x+2)^2 + (d+e*x)/(x^2+5);
```

$$FORM := \frac{a}{x-1}+\frac{b}{x+2}+\frac{c}{(x+2)^2}+\frac{d+e\,x}{x^2+5}$$

The second step is to determine the (linear) equations that define the constants `a`, `b`, `cc`, `d`, and `e`.

```
> eqn := f = FORM;
```

$$eqn := \frac{x^2-26\,x-47}{x^5+5\,x^3+3\,x^4+11\,x^2-20}=\frac{a}{x-1}+\frac{b}{x+2}+\frac{c}{(x+2)^2}+\frac{d+e\,x}{x^2+5}$$

```
> simplify( " );
```

$$\frac{x^2-26\,x-47}{x^5+5\,x^3+3\,x^4+11\,x^2-20}=(a\,x^4+4\,a\,x^3+9\,a\,x^2+20\,a\,x+20\,a+b\,x^4+b\,x^3+3\,b\,x^2+5\,b\,x-10\,b+c\,x^3-c\,x^2+5\,c\,x-5\,c+d\,x^3+3\,d\,x^2-4\,d+e\,x^4+3\,e\,x^3-4\,e\,x)\Big/\left((x-1)(x^2+5)(x+2)^2\right)$$

```
> collect( ", x );
```

$$\frac{x^2-26\,x-47}{x^5+5\,x^3+3\,x^4+11\,x^2-20}=((a+e+b)\,x^4+(4\,a+3\,e+c+d+b)\,x^3+(9\,a+3\,b-c+3\,d)\,x^2+(-4\,e+5\,b+5\,c+20\,a)\,x-5\,c-10\,b-4\,d+20\,a)\Big/\left((x-1)(x^2+5)(x+2)^2\right)$$

This equation will be satisfied (for all x, except 1 and -2) for any solution to the following system of five equations with five unknowns.

```
> eq1 := a+b+e=0;
> eq2 := 4*a+b+c+d+3*e;
> eq3 := 9*a+3*b-c+3*d = 1;
> eq4 := -4*e+5*b+5*c+20*a=-26;
> eq5 := -5*c-10*b-4*d+20*a = -47;
```

$$eq1 := a+e+b=0$$

$$eq2 := 4\,a+3\,e+c+d+b$$

$$eq3 := 9\,a + 3\,b - c + 3\,d = 1$$

$$eq4 := -4\,e + 5\,b + 5\,c + 20\,a = -26$$

$$eq5 := -5\,c - 10\,b - 4\,d + 20\,a = -47$$

The solution to these equations is:

```
> COEFFS := solve( {eq1,eq2,eq3,eq4,eq5}, {a,b,c,d,e} );
```

$$COEFFS := \left\{ e = \frac{13}{27}, b = \frac{23}{27}, d = \frac{91}{27}, c = \frac{-1}{3}, a = \frac{-4}{3} \right\}$$

Substituting this result, which is a set, into the original form for the partial fraction decomposition yields

```
> subs( COEFFS, FORM );
```

$$-\frac{4}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{1}{(x+2)^2} + \frac{23}{27}\,\frac{1}{x+2} + \frac{\frac{91}{27} + \frac{13}{27}\,x}{x^2+5}$$

Let's confirm the answer

```
> simplify( " );
```

$$\frac{x^2 - 26\,x - 47}{(x-1)\,(x^2+5)\,(x+2)^2}$$

The integral is now much simpler to evaluate by hand, but here it is one last time according to Maple.

```
> int( "", x );
```

$$-\frac{4}{3}\ln(x-1) + \frac{1}{3}\,\frac{1}{x+2} + \frac{23}{27}\ln(x+2) + \frac{13}{54}\ln(27\,x^2+135) + \frac{91}{135}\,\sqrt{5}\arctan\left(\frac{1}{5}\,x\,\sqrt{5}\right)$$

Note that this same outline can be used for ANY partial fraction problem. The key steps are i) knowing the correct form for the decomposition and ii) understanding how to identify the conditions that the coefficients must satisfy. These are (relatively) high level concepts; the user is relieved of the rather complicated mechanic manipulations.