Boundary Value Problems (§4.2)

BOUNDARY VALUE PROBLEMS (§4.2, #30, p. 159)

```
> restart;
> with( plots ):
> with( DEtools ):
```

Boundary value problems (BVPs) are another type of problem that is encountered in a number of applications of differential equations. For example, the temperature in a bar which has its ends held at fixed temperatures. A BVP is a differential equation with values of the solution specified at two different points. These points are often the endpoints of the interval on which the solution is needed. Thus, the asymptotic behavior of solutions for large values of the independent variable is not of primary interest. The questions of existence and uniqueness of solutions are more interesting for BVPs.

Objectives:

- to be exposed to BVPs as an alternative to IVPs
- to experience the different existence and uniqueness properties that can be found in BVPs
- to learn techniques for the use of Maple in the solution of BVPs

New Maple Commands:

`unapply` – converts an expression into a function

Problem 30 (Section 4.2, p. 159) Boundary Value Problems

When the values of a solution to a differential equation are specified at two different points, these conditions are called boundary conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point.)

In this problem we will consider the second-order differential equation

```
> ODE := diff( y(x), x$2 ) + y(x) = 0;
```

$$ODE := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right) + \mathrm{y}(x) = 0$$

which has the general solution

```
> GSOLN := dsolve( ODE, y(x) );
```

$$GSOLN := \mathrm{y}(x) = \_C1\cos(x) + \_C2\sin(x)$$

Show that

(a) there is a unique solution to the ODE that satisfies the boundary conditions

```
> BCa := y(0) = 2, y(Pi/2)=0;
```

$$BCa := \mathrm{y}(0) = 2, \mathrm{y}\left(\frac{1}{2}\pi\right) = 0$$

There are two different ways in which Maple can be used to solve this problem. The first, and simplest, is to use `dsolve` with the boundary conditions replacing the initial conditions. For example

```
> SOLNa := dsolve( { ODE, BCa }, y(x) );
```

$$SOLNa := \mathrm{y}(x) = 2\cos(x)$$

But, how do we know that this is the only solution to the problem? To definitively answer this question, it's necessary to work directly in terms of basic principles. All solutions to the differential equation must be in the form of the general solution. Showing that there is a unique solution to the BVP means that there are unique values of the constants `_C1` and `_C2` that force the general solution to satisfy the boundary conditions. To see how the boundary conditions place restrictions on the constants, it will be most convenient to have the general solution in the form of a Maple function (not as an equation or expression). The `unapply` command will do the necessary conversion

```
> Y := unapply( rhs(GSOLN), x );
```

$$Y := x \to \_C1\cos(x) + \_C2\sin(x)$$

Now, substituting this function into the boundary conditions yields

```
> CEQN := simplify( subs( y=Y, { BCa } ) );
```

$$CEQN := \{\_C1 = 2, \_C2 = 0\}$$

from which it is apparent that there is exactly one set of constants for which the boundary conditions are satisfied. These values of the constants identify the unique solution to the BVP as

```
> SOLNa2 := subs( CEQN, GSOLN );
```

$$SOLNa2 := \mathrm{y}(x) = 2\cos(x)$$

(Note the agreement between this solution and the one returned by `dsolve`.)

```
> SOLNa;
```

$$\mathrm{y}(x) = 2\cos(x)$$

(b) there is no solution to the ODE that satisfies

```
> BCb := y(0)=2, y(Pi)=0;
```

$$BCb := \mathrm{y}(0) = 2, \mathrm{y}(\pi) = 0$$

The same two methods can be used to solve this BVP. Because we are interested in the how and why of this example, the `dsolve` solution will not be displayed. As before, the first step is to substitute the general solution into the boundary conditions. The result is

```
> CEQN := simplify( subs( y=Y, { BCb } ) );
```

$$CEQN := \{-\_C1 = 0, \_C1 = 2\}$$

Note that the equations are inconsistent (`_C1` cannot be both 0 and 2). Thus, there is no choice of the constants for which the boundary conditions are satisfied and this BVP has no solution.

(c) there are infinitely many solutions to the ODE that satisfy

```
> BCc := y(0)=2, y(Pi)=-2;
```

$$BCc := \mathrm{y}(0) = 2, \mathrm{y}(\pi) = -2$$

The routine should now be familiar. Substitute the general solution into the boundary conditions, then examine the resulting equations for the constants `_C1` and `_C2`. In this case these yield

```
> CEQN := simplify( subs( y=Y, { BCc } ) );
```

$$CEQN := \{ \_C1 = 2, -\_C1 = -2 \}$$

These equations specify only one of the constants; `_C2` is still arbitrary.

```
> CSOLN := solve( CEQN, { _C1, _C2 } );
```

$$CSOLN := \{ \_C1 = 2, \_C2 = \_C2 \}$$

For each value of `_C2` a different solution to the boundary value problem is obtained.

```
> SOLNc := subs( CSOLN, GSOLN );
```

$$SOLNc := \mathrm{y}(x) = 2\cos(x) + \_C2\sin(x)$$

The same solutions are obtained using `dsolve`.

```
> SOLNc2 := dsolve( { ODE, BCc }, y(x) );
```

$$SOLNc2 := \mathrm{y}(x) = 2\cos(x) + \_C2\sin(x)$$

In either case, there are an infinite number of solutions to this problem.

These different types of solutions are very common in BVPs. In general, there is no way to look at a boundary value problem and decide how many solutions a BVP will have. (All that can be said is that the number will be either 0, 1, or ∞.)

To conclude this worksheet, let's consider a slight extension of this problem. The boundary conditions may involve more than the value of the function. Common examples are the value of the derivative of the function and a linear combination of the function and its first derivative at a point. Examples of each of these will be considered at this time.

EXAMPLE 1: For what values of α does the BVP

```
> restart;
> ODE := diff( y(x), x$2 ) - 6*diff( y(x), x ) + 5*y(x) = 1 - 2*x;
```

$$ODE := \left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right) - 6\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + 5\,\mathrm{y}(x) = 1 - 2x$$

```
> BC := y(0)=0, D(y)(2)=alpha;
```

$$BC := \mathrm{y}(0) = 0, \mathrm{D}(y)(2) = \alpha$$

have a unique solution. Plot this solution, and verify that the solution does satisfy the appropriate boundary conditions.

```
> dsolve( { ODE, BC }, y(x) );
```

$$\mathrm{y}(x) = -\frac{7}{25} - \frac{2}{5}x + \frac{1}{5}\frac{(7\,\mathrm{e}^{10} - 2 - 5\,\alpha)\,\mathrm{e}^{x}}{-\mathrm{e}^{2} + 5\,\mathrm{e}^{10}} + \frac{1}{25}\frac{(10 - 7\,\mathrm{e}^{2} + 25\,\alpha)\,\mathrm{e}^{(5\,x)}}{-\mathrm{e}^{2} + 5\,\mathrm{e}^{10}}$$

The general solution to this equation is easily found by hand, or with Maple.

```
> GSOLN := dsolve( ODE, y(x) );
```

$$GSOLN := \mathrm{y}(x) = -\frac{7}{25} - \frac{2}{5}x + \_C1\,\mathrm{e}^{x} + \_C2\,\mathrm{e}^{(5\,x)}$$

Preparing for later use, the general solution is converted from an equation to a function.

```
> Y := unapply( rhs(GSOLN), x );
```

$$Y := x \to -\frac{7}{25} - \frac{2}{5}x + \_C1\,\mathrm{e}^{x} + \_C2\,\mathrm{e}^{(5\,x)}$$

Equations that the constants must satisfy are determined by substituting the general solution into the boundary conditions.

```
> CEQN := simplify( subs( y=Y, { BC } ) );
```

$$CEQN := \left\{ -\frac{2}{5} + \_C1\,\mathrm{e}^2 + 5\,\_C2\,\mathrm{e}^{10} = \alpha, -\frac{7}{25} + \_C1 + \_C2 = 0 \right\}$$

While these equations are slightly more complicated than those encountered earlier in this worksheet, note that there are two linear equations for the two constants. The solution to this system is

```
> CSOLN := solve( CEQN, { _C1, _C2 } );
```

$$CSOLN := \left\{ \_C2 = \frac{1}{25}\,\frac{10 - 7\,\mathrm{e}^2 + 25\,\alpha}{-\mathrm{e}^2 + 5\,\mathrm{e}^{10}}, \_C1 = \frac{1}{5}\,\frac{7\,\mathrm{e}^{10} - 2 - 5\,\alpha}{-\mathrm{e}^2 + 5\,\mathrm{e}^{10}} \right\}$$

Note that these solutions are defined for all values of α, and that the solution is unique. Thus, the BVP has a unique solution for all values of α. These solutions are

```
> SOLN := simplify( subs( CSOLN, GSOLN ) );
```

$$SOLN := \mathrm{y}(\,x\,) = -\frac{1}{25}(-7\,\mathrm{e}^2 + 35\,\mathrm{e}^{10} - 10\,x\,\mathrm{e}^2 + 50\,x\,\mathrm{e}^{10} - 35\,\mathrm{e}^{(\,x+10\,)} + 10\,\mathrm{e}^x + 25\,\mathrm{e}^x\,\alpha$$
$$- 10\,\mathrm{e}^{(\,5\,x\,)} + 7\,\mathrm{e}^{(\,5\,x+2\,)} - 25\,\mathrm{e}^{(\,5\,x\,)}\,\alpha) \Big/ (\,-\mathrm{e}^2 + 5\,\mathrm{e}^{10}\,)$$

Using **dsolve** to solve this problem leads to

```
> SOLN2 := dsolve( { ODE, BC }, y(x) );
```

$$SOLN2 := \mathrm{y}(\,x\,) = -\frac{7}{25} - \frac{2}{5}\,x + \frac{1}{5}\,\frac{(\,7\,\mathrm{e}^{10} - 2 - 5\,\alpha\,)\,\mathrm{e}^x}{-\mathrm{e}^2 + 5\,\mathrm{e}^{10}} + \frac{1}{25}\,\frac{(\,10 - 7\,\mathrm{e}^2 + 25\,\alpha\,)\,\mathrm{e}^{(\,5\,x\,)}}{-\mathrm{e}^2 + 5\,\mathrm{e}^{10}}$$

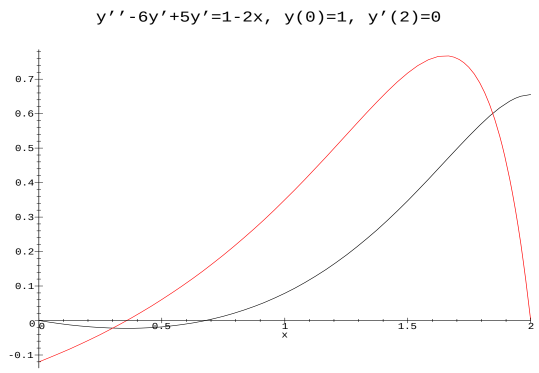
The two solutions are equivalent, as can be easily verified

```
> simplify( rhs(SOLN) - rhs(SOLN2) );
```

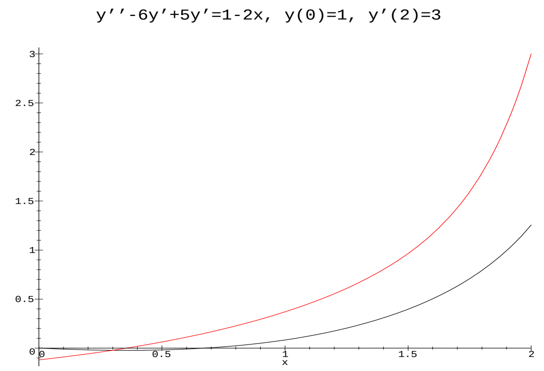
$$0$$

To conclude this example, it is most appropriate to plot both the solution and its first derivative on the same axes

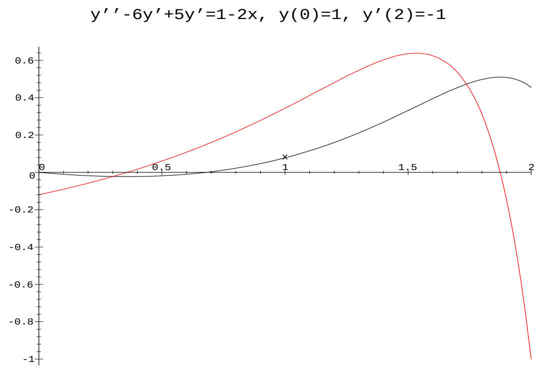
```
> plot( subs( alpha=0, { rhs(SOLN), diff( rhs(SOLN), x ) }), x=0..2,
>       title=`y''-6y'+5y'=1-2x, y(0)=1, y'(2)=0` );
```



```
> plot( subs( alpha=3, { rhs(SOLN), diff( rhs(SOLN), x ) }), x=0..2,
>       title=`y''-6y'+5y'=1-2x, y(0)=1, y'(2)=3` );
```




```
> plot( subs( alpha=-1, { rhs(SOLN), diff( rhs(SOLN), x ) }), x=0..2,
>       title=`y''-6y'+5y'=1-2x, y(0)=1, y'(2)=-1` );
```



EXAMPLE 2: For what values of the parameter α does the following boundary value problem have a solution?

```
> restart;
> ODE := diff( y(x), x$2 ) + 4*diff( y(x), x ) + 4*y(x) = exp(-2*x);
```

$$ODE := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right) + 4\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) + 4\,\mathrm{y}(x) = \mathrm{e}^{(-2\,x)}$$

```
> BC := y(0)=alpha, D(y)(Pi) - y(Pi) = 0;
```

$$BC := \mathrm{y}(0) = \alpha, \mathrm{D}(y)(\pi) - \mathrm{y}(\pi) = 0$$

Note the new form of the boundary condition at $x = \pi$. This type of boundary condition cannot be fed to `dsolve`.

```
> dsolve( { ODE, BC }, y(x) );
Error, (in dsolve) invalid arguments
```

Thus, our only hope (using Maple) is the long method introduced previously in this worksheet. The first step is to determine the general solution for the nonhomogeneous equation

```
> GSOLN := dsolve( ODE, y(x) );
```

$$GSOLN := \mathrm{y}(x) = \frac{1}{2}\,x^2\,\mathrm{e}^{(-2\,x)} + \_C1\,\mathrm{e}^{(-2\,x)} + \_C2\,\mathrm{e}^{(-2\,x)}\,x$$

```
> Y := unapply( rhs(GSOLN), x );
```

$$Y := x \to \frac{1}{2}\,x^2\,\mathrm{e}^{(-2\,x)} + \_C1\,\mathrm{e}^{(-2\,x)} + \_C2\,\mathrm{e}^{(-2\,x)}\,x$$

The constraints to be satisfied by the constants are

```
> CEQN := simplify( subs( y=Y, { BC } ) );
```

$$CEQN := \left\{\pi\,\mathrm{e}^{(-2\,\pi)} - \frac{3}{2}\,\pi^2\,\mathrm{e}^{(-2\,\pi)} - 3\,\_C1\,\mathrm{e}^{(-2\,\pi)} - 3\,\_C2\,\mathrm{e}^{(-2\,\pi)}\,\pi + \_C2\,\mathrm{e}^{(-2\,\pi)} = 0, \_C1 = \alpha\right\}$$

The values of the constants that satisfy these conditions are

```
> CSOLN := solve( CEQN, { _C1, _C2 } );
```

$$CSOLN := \left\{\_C1 = \alpha, \_C2 = -\frac{1}{2}\,\frac{3\,\pi^2 - 2\,\pi + 6\,\alpha}{-1 + 3\,\pi}\right\}$$

It is now apparent that they are well-defined for all values of α – hence the BVP has a unique solution for all α. This solution is

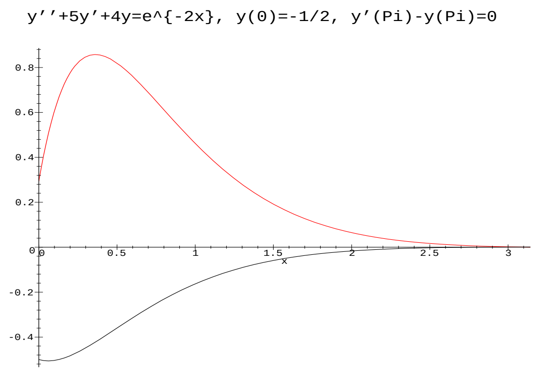
```
> SOLN := subs( CSOLN, GSOLN );
```

$$SOLN := \mathrm{y}(x) = \frac{1}{2}\,x^2\,\mathrm{e}^{(-2\,x)} + \alpha\,\mathrm{e}^{(-2\,x)} - \frac{1}{2}\,\frac{\left(3\,\pi^2 - 2\,\pi + 6\,\alpha\right)\,\mathrm{e}^{(-2\,x)}\,x}{-1 + 3\,\pi}$$

The solution that corresponds with $\alpha = 1$ is

```
> YY := subs( alpha=-1/2, rhs(SOLN) );
> plot( { YY, diff( YY, x ) - YY }, x=0..Pi,
>       title=`y''+5y'+4y=e^{-2x}, y(0)=-1/2, y'(Pi)-y(Pi)=0` );
```

$$YY := \frac{1}{2}\,x^2\,\mathrm{e}^{(-2\,x)} - \frac{1}{2}\,\mathrm{e}^{(-2\,x)} - \frac{1}{2}\,\frac{\left(3\,\pi^2 - 2\,\pi - 3\right)\,\mathrm{e}^{(-2\,x)}\,x}{-1 + 3\,\pi}$$



It can also be interesting, and informative, to animate these solutions using α as the animation parameter.

```
> YYY := unapply( rhs(SOLN), (x,alpha) );
```

$$YYY := (\,x, \alpha\,) \to \frac{1}{2}\,x^2\,\mathrm{e}^{(\,-2\,x\,)} + \alpha\,\mathrm{e}^{(\,-2\,x\,)} - \frac{1}{2}\,\frac{\left(3\,\pi^2 - 2\,\pi + 6\,\alpha\right)\,\mathrm{e}^{(\,-2\,x\,)}\,x}{-1 + 3\,\pi}$$

```
> plots[animate](YYY(x,alpha), x=0..Pi, alpha=-1..1 );
```

From these plots, it appears as though $y(\pi) = 0$ for all values of α. Is this true? (Explain.)