>

Laplace Transforms

laplace.mws

Douglas B. Meade
(meade@math.sc.edu)

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Overview

The Laplace transform is a very common, and useful, technique for solving and analyzing the solution of initial value problems for linear and constant coefficient equations and systems. The key property of the Laplace transform is that derivatives are transformed into algebraic powers of the transform variable; thus, the differential equation is transformed into an algebraic equation. The focus of this worksheet is to understand the Laplace transform, and the inverse Laplace transform, more than the actual solution of a differential equation.

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Objectives

- to understand the definition of the Laplace transform and the inverse Laplace transform
- to appreciate the general applicability of Laplace transforms to the solution of linear ODEs (both single equations and systems)
- to understand and be able to work with piecewise defined functions
- to become familiar with several of the different ways in which Maple can deal with Laplace transforms

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New Maple Commands

- alias — create an alternate name by which an expression can be referenced
- completesquare — complete the square of a polynomial of degree 2 (in student package)
- convert(..., parfrac, s) — compute the partial fraction decomposition of an expression
- convert(..., piecewise) — convert from Heaviside functions to piecewise-defined functions
- dsolve(..., method=laplace) — use Laplace transforms to solve a differential equation
- Heaviside — the Heaviside step function (zero if x<0, one if x>0, and undefined for x=0)
- invlaplace — apply the inverse Laplace transform (in inttrans package)
- laplace — apply the Laplace transform (in inttrans package)
- piecewise — create a piecewise-defined "function"

>

Optional

- about — check assumptions for a variable
- assume — provide additional information or constraints on a variable

>

Preliminary Preparations

```
> restart;
>
```

The full implementation of the Laplace transform is available only after the inttrans library is loaded.

```
> with( inttrans );
```

$$[addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, laplace, mellin]$$

```
>
```

Likewise, the student package is needed if it is desired to complete the square on an expression (using the completesquare command).

```
> with( student );
```

$$[D, Diff, Doubleint, Int, Limit, Lineint, Product, Sum, Tripleint, changevar, combine, completesquare, distance, equate, extrema, integrand, intercept, intparts, isolate, leftbox, leftsum, makeproc, maximize, middlebox, middlesum, midpoint, minimize, powsubs, rightbox, rightsum, showtangent, simpson, slope, trapezoid, value]$$

```
>
```

The Laplace Transform

Definition 1: The Laplace transform of a function f is defined to be the function F given by the improper integral

$$F(s) = \int_0^\infty f(t)\, e^{(-s\,t)}\, dt$$

for all values of s for which the integral exists.

```
>
```

Examples using the definition

The definition of the Laplace transform can be used to find the transform of

the constant function $f(t) = 1$

Solution

```
> LT[1] := Int( 1 * exp(-s*t), t=0..infinity ):
> LT[1] = value( LT[1] );
Definite integration: Can't determine if the integral is convergent.
Need to know the sign of --> s
Will now try indefinite integration and then take limits.
```

$$\int_0^\infty e^{(-s\,t)}\, dt = \lim_{t \to \infty} -\frac{e^{(-s\,t)}}{s} + \frac{1}{s}$$

```
>
```

Note that this result -- in particular, the limit term -- is not what you will find in a table of Laplace transforms. Note, however, that Maple has provided additional information that indicates additional information that might enable it to simplify this answer.

```
>
```

New Feature in Release 4

The messages reporting the information that Maple needs to determine if the integral is convergent is new to Release 4. Earlier releases simply returned the result with the limit.

```
>
```

Why is the sign of s important in this problem?

If s<0 then the exponential grows without bound as t increases. If s=0 then the integrand simplifies to 1. In either case, the improper integral diverges; hence, the Laplace transform can only be defined for s >0.

```
>
```

Assumptions -- optional

Maple provides a mechanism, known as the assume facility, by which user's can specify additional information about a variable. While this is beyond the scope of the current discussion, interested students might appreciate the following steps that can be taken obtain a cleaner answer from the definition of the Laplace transform.

```
> assume( s>0 );
> about( s );
> LT[1] = value( LT[1] );
```

$$\int_0^{\infty} e^{(-s\sim t)}\, dt = \frac{1}{s\sim}$$

```
>
```

This is what we would expect to see as the Laplace transform of 1.

Assumptions should always be removed at the end of a problem. The simplest way to clear all assumptions about a name is to simply reassign the name to it's unevaluated state. That is,

```
> s := 's';
```

$$s := s$$

```
> about( s );
>
```

Explanation of tilde in output

The tilde (~) that appears after the s is included as an indication that additional information has been provided about this name.

```
>
```

the exponential function $f(t) = e^{(a\,t)}$

```
> LT[exp] := Int( exp(a*t) * exp(-s*t), t=0..infinity ):
> LT[exp] = value( LT[exp] );
Definite integration: Can't determine if the integral is convergent.
Need to know the sign of --> -a+s
Will now try indefinite integration and then take limits.
```

$$\int_0^{\infty} e^{(a\,t)}\, e^{(-s\,t)}\, dt = \lim_{t\to\infty} \frac{e^{(a\,t - s\,t)}}{a-s} - \frac{1}{a-s}$$

```
>
```

Once again Maple needs some additional information. Using the same approach as in the first example, it is not difficult to see that the above limit exists only when $s > a$. Then, the exponential decays to zero in the limit and the value of the improper integral is $\frac{1}{s-a}$, as expected.

```
>
```

the trigonometric function $f(t) = \cos(a\,t)$

```
> LT[cos] := Int( cos(a*t)*exp(-s*t), t=0..infinity ):
> LT[cos] = value( LT[cos] );
```

$$\int_0^{\infty} \cos(a\,t)\, e^{(-s\,t)}\, dt = \lim_{t\to\infty} -\frac{s\, e^{(-s\,t)} \cos(a\,t)}{s^2+a^2} + \frac{a\, e^{(-s\,t)} \sin(a\,t)}{s^2+a^2} + \frac{s}{s^2+a^2}$$

Clarification of Notation

Note that the limit applies to the first two terms in the above expression. Ideally, Maple should have included parentheses to clearly indicate that the limit is applied to more than the first term.

```
>
```

Maple does not provide any hints as to the information that it needs to complete the evaluation of the improper integral. However, based on the previous two examples, it seems that -- at the least -- we will need to assume $s > 0$. In fact, under exactly this assumption, each of the first two terms decays to zero and the Laplace transform is the last term: $\frac{s}{s^2+a^2}$.

```
>
>
```

Examples using the laplace command

The laplace command automatically assumes the appropriate conditions on the transform variable. While this does simplify the determination of Laplace transforms, it is the user's responsibility to ensure that the results are used appropriately.

```
>
```

The Laplace transforms of the three examples could have obtained as follows:

```
> laplace( 1, t, s );
```

$$\frac{1}{s}$$

```
> laplace( exp(a*t), t, s );
```

$$\frac{1}{-a+s}$$

```
> laplace( cos(a*t), t, s );
```

$$\frac{s}{s^2+a^2}$$

```
>
```

The Inverse Laplace Transform

Inverse Laplace transform works similarly, except that there is no (useful) explicit formula. Regardless, Maple can still be quite useful for the computation of inverse Laplace transforms. Note that many of the functions that appear as Laplace transforms in a typical Table of Laplace Transforms involve terms involving i) a power of s, ii) a linear factor involving s, or iii) an irreducible quadratic expression in s. The key to the successful determination of inverse Laplace transforms is the ability to convert an expression into a form that can be inverted using the Table of Laplace Transforms. The convert(..., parfrac, s) and completesquare commands will be seen t be particularly helpful in completing this task.

```
>
```

Example 1

Find the inverse Laplace transform of $F_1(s) = \dfrac{7\,s-1}{s^3-7\,s-6}$.

```
>
```

Solution

```
> F[1] := (7*s-1)/(s^3-7*s-6);
```

$$F_1 := \frac{7\,s-1}{s^3-7\,s-6}$$

It is not difficult to see that F_1 has at least one real root ($s=-1$). The full partial fraction decomposition of F_1 is:

```
> convert( F[1], parfrac, s );
```

$$\frac{1}{s-3} - \frac{3}{s+2} + \frac{2}{s+1}$$

```
>
```

The inverse Laplace transform of each term is easily determined: $\mathbf{e}^{(3\,t)}$, $-3\,\mathbf{e}^{(-2\,t)}$, and $2\,\mathbf{e}^{(-t)}$. These results can be confirmed by comparing with the results of the invlaplace command:

```
> f[1] := invlaplace( F[1], s, t );
```

$$f_1 := \mathbf{e}^{(3\,t)} - 3\,\mathbf{e}^{(-2\,t)} + 2\,\mathbf{e}^{(-t)}$$

```
>
```

Example 2

Find the inverse Laplace transform of $F_2(s) = \dfrac{s-1}{s^2-2\,s+5}$.

```
>
```

Solution

```
> F[2] := (s-1)/(s^2-2*s+5);
```

$$F_2 := \frac{s-1}{s^2-2\,s+5}$$

Since the denominator is an irreducible quadratic, it will be fruitless to look for a partial fraction decomposition of F_2. Instead, what is needed is to "complete the square" in the denominator of F_2. This is achieved by using the completesquare command:

```
> completesquare( F[2] );
```

$$\frac{s-1}{(s-1)^2+4}$$

```
>
```

The presence of $(s-1)$ says that the inverse Laplace transform includes $\mathbf{e}^t$. The inverse Laplace transform of $\frac{s}{s^2+4}$ is found in the table to be $\cos(2\,t)$; thus, the inverse Laplace transform of F_2 must be $f_2(t)=\mathbf{e}^t\cos(2\,t)$.
As before, this answer is easily checked using Maple

```
> f[2] := invlaplace( F[2], s, t );
```

$$f_2 := \mathbf{e}^t\cos(2\,t)$$

```
>
```

Example 3

Find the inverse Laplace transform of $F_3(s)=\frac{\mathbf{e}^{(-2\,s)}}{s^3}$.

```
>
```

Solution

While this example looks innocuous enough, the exponential term presents some new twists. The relevant entries in a table are more complicated than we have encountered in prior examples, let's forego the hand computations and simply ask Maple for the inverse Laplace transform.

```
> F[3] := exp(-2*s)/s^3;
```

$$F_3 := \frac{\mathbf{e}^{(-2\,s)}}{s^3}$$

```
> f[3] := invlaplace( F[3], s, t );
```

$$f_3 := \frac{1}{2}\,\mathrm{Heaviside}(t-2)\,t^2-2\,\mathrm{Heaviside}(t-2)\,t+2\,\mathrm{Heaviside}(t-2)$$

```
> f[3] := factor( f[3] );
```

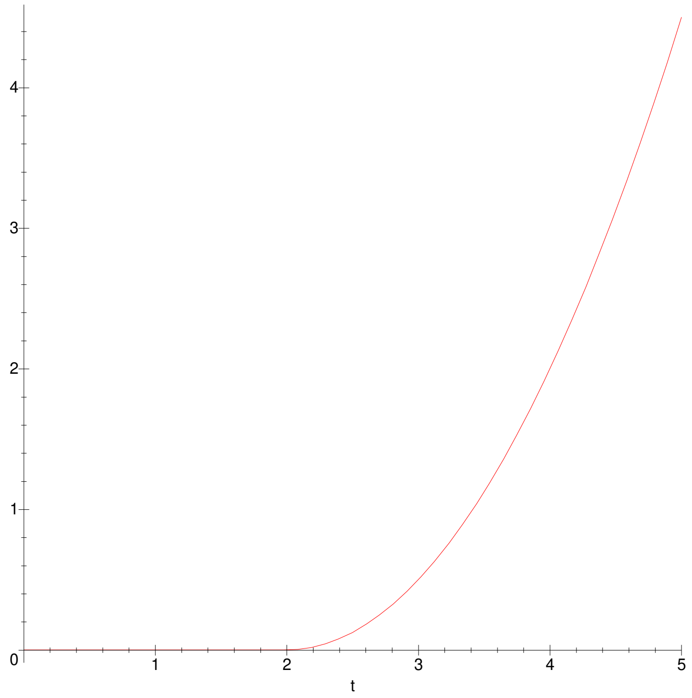
$$f_3 := \frac{1}{2}\,\mathrm{Heaviside}(t-2)\,(t-2)^2$$

```
>
```

What is this function Heaviside? One of the easiest ways to learn about Heaviside functions is to plot this function, and attempt to understand how it relates to this function.

```
> plot( f[3], t=0..5, title=`Function with Heaviside` );
```

Function with Heaviside



>

Notice that the solution appears to be identically zero for t<2, and is quadratic with a vertex at t=2 for t>2. The quadratic behavior is seen in the term, $\frac{(t-2)^2}{2}$, that multiplies the Heaviside function. Since this is not seen for t<2, it appears as though $\mathrm{Heaviside}(t-2)$ must be identically zero for t<2. (How might you use Maple to test this conjecture?)

>

Update for Release 4

A new feature in Release 4 is the ability to convert between Heaviside functions and piecewise-defined functions (see ?convert,piecewise). In this case, we find

```
> convert( f[3], piecewise );
```

$$\begin{cases} 0 & t \leq 2 \\ \frac{1}{2}t^2-2t+2 & 2<t \end{cases}$$

This is an even clearer illustration of the properties observed from the plot.

>

In conclusion, the inverse Laplace transform of F_3 is identically zero for t<=2 as a result of the coefficient 2 in the exponential and; for t>2, the presence of the cubic power in the denominator of F_3 is the source of the quadratic term in f_3.

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Laplace Transforms for Initial Value Problems

This is the type of problem that is typically of most interest when studying Laplace transforms, particularly in a differential equations course..

>

Example 1

To illustrate the general procedure for solving an IVP using Laplace transforms, let's use the Laplace transform to find the solution to the initial-value problem:

$$y'' - 3\,y' + 2\,y = \cos(t) + (t^2 - 3)\,\mathbf{e}^{(-2\,t)},\quad y(0) = 0,\quad y'(0)=1.$$

\>

Solution

The IVP can be described with:

```
> ODE := diff( y(t), t$2 ) - 3*diff( y(t), t ) + 2*y(t)
>        = cos(t) + (t^2-3) * exp(-2*t);
> IC := y(0) = 0, D(y)(0) = 1;
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\,\mathrm{y}(t)\right) - 3\left(\frac{\partial}{\partial t}\,\mathrm{y}(t)\right) + 2\,\mathrm{y}(t) = \cos(t) + (t^2 - 3)\,\mathbf{e}^{(-2\,t)}$$

$$IC := \mathrm{y}(0) = 0,\ \mathrm{D}(y)(0) = 1$$

\>

The first step is to apply the Laplace transform to the differential equation:

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := s\,(s\,\mathrm{laplace}(\mathrm{y}(t), t, s) - \mathrm{y}(0)) - \mathrm{D}(y)(0) - 3\,s\,\mathrm{laplace}(\mathrm{y}(t), t, s) + 3\,\mathrm{y}(0) + 2\,\mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{s}{s^2+1} + \frac{2}{(s+2)^3} - \frac{3}{s+2}$$

\>

Note the way in which the Laplace transform of the solution is denoted in this equation: laplace(y(t), t, s). Note also the presence of y(0) and D(y)(0) in the transformed equation. Since these values are known from the initial conditions, they should be introduced at the earliest convenience. For example:

```
> ODE_lap2 := subs( IC, ODE_lap );
```

$$ODE\_lap2 := s^2\,\mathrm{laplace}(\mathrm{y}(t), t, s) - 1 - 3\,s\,\mathrm{laplace}(\mathrm{y}(t), t, s) + 2\,\mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{s}{s^2+1} + \frac{2}{(s+2)^3} - \frac{3}{s+2}$$

Next, it is necessary to explicitly solve for the Laplace transform of the solution:

```
> SOLN_lap := solve( ODE_lap2, { laplace( y(t), t, s ) } );
```

$$SOLN\_lap := \{\mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{8\,s - 2 + 13\,s^2 + s^5 + 4\,s^4 + 7\,s^3}{16 + 3\,s^6 + s^7 - 4\,s^3 - 13\,s^4 - 3\,s^5}\}$$

\>

The solution can be obtained in one step using invlaplace, but the solution is also available by inspection once the right-hand side is written in its partial fraction decomposition.

```
> SOLN_lap2 := convert( SOLN_lap, parfrac, s );
```

$$SOLN\_lap2 := \{\mathrm{laplace}(\mathrm{y}(t), t, s) = -\frac{31}{54}\,\frac{1}{s-1} + \frac{109}{160}\,\frac{1}{s-2} + \frac{1}{6}\,\frac{1}{(s+2)^3} + \frac{7}{72}\,\frac{1}{(s+2)^2} - \frac{179}{864}\,\frac{1}{s+2} + \frac{1}{10}\,\frac{s-3}{s^2+1}\}$$

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \{\mathrm{y}(t) = -\frac{31}{54}\,\mathbf{e}^t + \frac{109}{160}\,\mathbf{e}^{(2\,t)} + \frac{1}{12}\,\mathbf{e}^{(-2\,t)}\,t^2 + \frac{7}{72}\,t\,\mathbf{e}^{(-2\,t)} - \frac{179}{864}\,\mathbf{e}^{(-2\,t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\}$$

\>

Compare this solution with the solutions obtained by directly inverting the Laplace transform once it is available

```
> invlaplace( SOLN_lap, s, t );
```

$$\{\mathrm{y}(t) = -\frac{31}{54}\,\mathbf{e}^t + \frac{109}{160}\,\mathbf{e}^{(2\,t)} + \frac{1}{12}\,\mathbf{e}^{(-2\,t)}\,t^2 + \frac{7}{72}\,t\,\mathbf{e}^{(-2\,t)} - \frac{179}{864}\,\mathbf{e}^{(-2\,t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)\}$$

or the solution obtained by directly with dsolve if we tell Maple to use Laplace transforms to solve the problem.

```
> dsolve( { ODE, IC }, y(t), method=laplace );
```

$$\mathrm{y}(t) = -\frac{31}{54}\,\mathbf{e}^t + \frac{109}{160}\,\mathbf{e}^{(2\,t)} + \frac{1}{12}\,\mathbf{e}^{(-2\,t)}\,t^2 + \frac{7}{72}\,t\,\mathbf{e}^{(-2\,t)} - \frac{179}{864}\,\mathbf{e}^{(-2\,t)} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)$$

Point to Ponder

Identify the homogeneous and particular solutions in this solution

\>

\>

Example 2

Use Laplace transforms to find the general solution to $y'' + 2\,y = \cos(2\,t)$.

>

Solution

```
> ODE := diff( y(t), t$2 )  + 2*y(t) = cos(2*t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\mathrm{y}(t)\right) + 2\,\mathrm{y}(t) = \cos(2\,t)$$

>

The Laplace transform of this equation is

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := s\,(s\,\mathrm{laplace}(\mathrm{y}(t), t, s) - \mathrm{y}(0)) - \mathrm{D}(y)(0) + 2\,\mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{s}{s^2+4}$$

```
> SOLN_lap := solve( ODE_lap, { laplace(y(t),t,s) } );
```

$$SOLN\_lap := \{\mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{\mathrm{y}(0)\,s^3 + 4\,s\,\mathrm{y}(0) + \mathrm{D}(y)(0)\,s^2 + 4\,\mathrm{D}(y)(0) + s}{s^4 + 6\,s^2 + 8}\}$$

```
> invlaplace( SOLN_lap, s, t );
```

$$\{\mathrm{y}(t) = -\frac{1}{2}\cos(2\,t) + \frac{1}{2}\cos(\sqrt{2}\,t) + \mathrm{y}(0)\cos(\sqrt{2}\,t) + \frac{1}{2}\sqrt{2}\,\mathrm{D}(y)(0)\sin(\sqrt{2}\,t)\}$$

>

Note that the two constants in this solution are the two initial conditions at $t = 0$.

As before, there are several alternate methods to reach the same solution, including

```
> dsolve( ODE, y(t), method=laplace );
```

$$\mathrm{y}(t) = -\frac{1}{2}\cos(2\,t) + \frac{1}{2}\cos(\sqrt{2}\,t) + \mathrm{y}(0)\cos(\sqrt{2}\,t) + \frac{1}{2}\sqrt{2}\,\mathrm{D}(y)(0)\sin(\sqrt{2}\,t)$$

>

It is particularly noteworthy to compare this solution with the one returned by dsolve when the solution method is not explicitly specified:

```
> dsolve( ODE, y(t) );
```

$$\mathrm{y}(t) = -\frac{1}{4}\sin(\sqrt{2}\,t)\sin((\sqrt{2}-2)\,t) - \frac{1}{4}\sin(\sqrt{2}\,t)\sin((\sqrt{2}-2)\,t)\sqrt{2} - \frac{1}{4}\sin(\sqrt{2}\,t)\sin((2+\sqrt{2})\,t)$$
$$+\frac{1}{4}\sin(\sqrt{2}\,t)\sin((2+\sqrt{2})\,t)\sqrt{2} - \frac{1}{4}\cos(\sqrt{2}\,t)\cos((2+\sqrt{2})\,t) + \frac{1}{4}\cos(\sqrt{2}\,t)\cos((2+\sqrt{2})\,t)\sqrt{2}$$
$$-\frac{1}{4}\cos(\sqrt{2}\,t)\cos((\sqrt{2}-2)\,t) - \frac{1}{4}\cos(\sqrt{2}\,t)\cos((\sqrt{2}-2)\,t)\sqrt{2} + \_C1\sin(\sqrt{2}\,t) + \_C2\cos(\sqrt{2}\,t)$$

>

This type of solution has been observed previously (see forced.mws). The key in that situation was to ask Maple to apply a variety of trigonometric identities to convert the solution to a more standard form:

```
> combine( ", trig );
```

$$\mathrm{y}(t) = -\frac{1}{2}\cos(2\,t) + \_C1\sin(\sqrt{2}\,t) + \_C2\cos(\sqrt{2}\,t)$$

>

Point to Ponder

What is the relationship between the initial conditions y(0) and D(y)(0) and the constants _C1 and _C2?

>

Example 3

Find the general solution to the differential equation $y'' + 2\,y' = 3\,\mathrm{Dirac}(t-1) - \mathrm{Heaviside}(t-\pi)\sin(t) + \mathbf{e}^{(-t)}$.

>

Solution

The same techniques can be used to solve problems that involve piecewise-defined forcing functions (Heaviside) and/or impulse functions (Dirac).

```
> ODE := diff( y(t), t$2 ) + 2 * diff( y(t), t )
>        = 3 * Dirac(t-1) + Heaviside(t-Pi) * sin(t-Pi) + exp(-t);
```

$$ODE := \left(\frac{\partial^2}{\partial t^2}\,\mathrm{y}(t)\right) + 2\left(\frac{\partial}{\partial t}\,\mathrm{y}(t)\right) = 3\,\mathrm{Dirac}(t-1) - \mathrm{Heaviside}(t-\pi)\,\sin(t) + \mathbf{e}^{(-t)}$$

```
>
```

The Laplace transform of the equation is:

```
> ODE_lap := laplace( ODE, t, s );
```

$$ODE\_lap := s\,(s\,\mathrm{laplace}(\mathrm{y}(t), t, s) - \mathrm{y}(0)) - \mathrm{D}(y)(0) + 2\,s\,\mathrm{laplace}(\mathrm{y}(t), t, s) - 2\,\mathrm{y}(0) = 3\,\mathbf{e}^{(-s)} + \frac{\mathbf{e}^{(-s\,\pi)}}{s^2+1} + \frac{1}{s+1}$$

```
>
```

While we have managed to work with the long form for the Laplace transform of y, the alias command can be used to provide a simpler, and more natural, name to use for this quantity.

```
> alias( Y(s) = laplace( y(t), t, s ) ):
> ODE_lap;
```

$$s\,(s\,\mathrm{Y}(s) - \mathrm{y}(0)) - \mathrm{D}(y)(0) + 2\,s\,\mathrm{Y}(s) - 2\,\mathrm{y}(0) = 3\,\mathbf{e}^{(-s)} + \frac{\mathbf{e}^{(-s\,\pi)}}{s^2+1} + \frac{1}{s+1}$$

```
>
> SOLN_lap := solve( ODE_lap, { Y(s) } );
```

$$SOLN\_lap := \{\mathrm{Y}(s) = (1 + s^2 + \mathbf{e}^{(-s\,\pi)} + 3\,\mathbf{e}^{(-s)} + 3\,\mathrm{y}(0)\,s^2 + 3\,\mathrm{y}(0)\,s^3 + 3\,s\,\mathrm{y}(0) + \mathrm{y}(0)\,s^4 + \mathrm{D}(y)(0)\,s^3 + \mathrm{D}(y)(0)\,s^2$$
$$+ \mathrm{D}(y)(0)\,s + 3\,\mathbf{e}^{(-s)}\,s^2 + 3\,\mathbf{e}^{(-s)}\,s + \mathbf{e}^{(-s\,\pi)}\,s + 3\,\mathbf{e}^{(-s)}\,s^3 + 2\,\mathrm{y}(0) + \mathrm{D}(y)(0)) \,/\, (s\,(2 + 3\,s^2 + 3\,s^3 + 3\,s + s^4))\}$$

```
> SOLN := invlaplace( SOLN_lap, s, t );
```

$$SOLN := \{\mathrm{y}(t) = \frac{1}{2}\,\mathrm{Heaviside}(t-\pi) + \frac{3}{2}\,\mathrm{Heaviside}(t-1) - \frac{1}{2}\,\mathrm{D}(y)(0)\,\mathbf{e}^{(-2\,t)} + \frac{1}{2}\,\mathbf{e}^{(-2\,t)} - \mathbf{e}^{(-t)} + \frac{1}{2} + \mathrm{y}(0) + \frac{1}{2}\,\mathrm{D}(y)(0)$$
$$+ \frac{1}{5}\,\mathrm{Heaviside}(t-\pi)\,\sin(t) + \frac{2}{5}\,\mathrm{Heaviside}(t-\pi)\,\cos(t) - \frac{1}{10}\,\mathrm{Heaviside}(t-\pi)\,\mathbf{e}^{(-2\,t+2\,\pi)} - \frac{3}{2}\,\mathrm{Heaviside}(t-1)\,\mathbf{e}^{(-2\,t+2)}\}$$

```
>
```

This solution look needlessly complicated; it can be written in a format that clearly identifies the role of the initial data and the different Heaviside functions that appear in the solution. The main terms in the solution are

```
> TERMS := { y(0), D(y)(0), Heaviside(t-1), Heaviside(t-Pi) };
```

$$TERMS := \{\mathrm{Heaviside}(t-1), \mathrm{y}(0), \mathrm{D}(y)(0), \mathrm{Heaviside}(t-\pi)\}$$

```
> SOLN := collect( op(SOLN), TERMS );
```

$$SOLN := \mathrm{y}(t) = \left(\frac{3}{2} - \frac{3}{2}\,\mathbf{e}^{(-2\,t+2)}\right)\mathrm{Heaviside}(t-1) + \mathrm{y}(0) + \left(-\frac{1}{2}\,\mathbf{e}^{(-2\,t)} + \frac{1}{2}\right)\mathrm{D}(y)(0)$$
$$+ \left(\frac{1}{2} + \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) - \frac{1}{10}\,\mathbf{e}^{(-2\,t+2\,\pi)}\right)\mathrm{Heaviside}(t-\pi) + \frac{1}{2} + \frac{1}{2}\,\mathbf{e}^{(-2\,t)} - \mathbf{e}^{(-t)}$$

```
>
```

A plot of the solution with homogeneous initial conditions

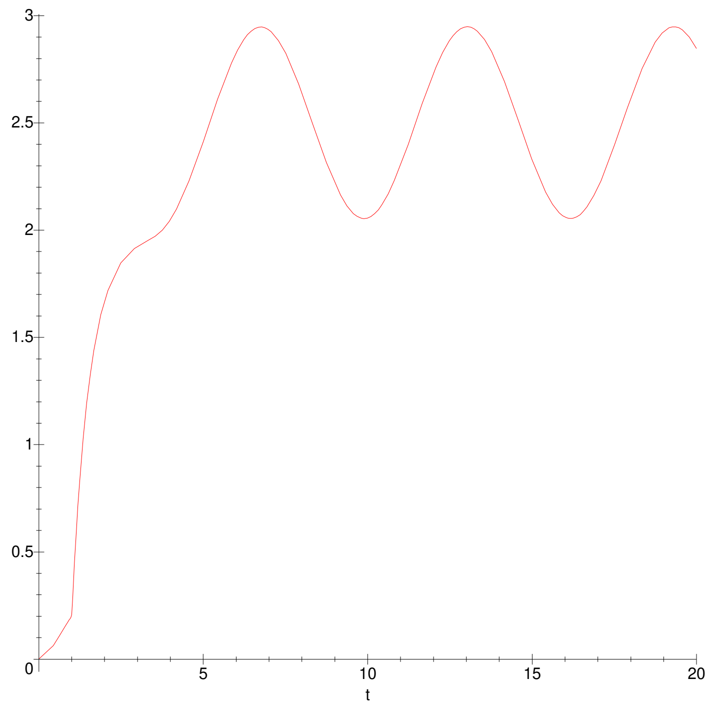
```
> IC := y(0)=0, D(y)(0)=0;
```

$$IC := \mathrm{y}(0) = 0, \mathrm{D}(y)(0) = 0$$

is also illuminating

```
> plot( subs( IC, rhs(SOLN) ), t=0..20, title=`Solution to IVP with Heaviside` );
```

Solution to IVP with Heaviside



Notice that the solution appears to undergo changes in form at $t = 1$ and $t = \pi$.

>

Points to Ponder

- What does the solution look like as a piecewise-defined function? (Do this for both general and homogeneous initial conditions.)
- Is the same solution can be obtained directly using dsolve with method=laplace?
- What if method=laplace is omitted?

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Laplace Transforms for Systems of Linear ODEs

One of the great strengths of the Laplace transform is its applicability to both single equations and systems using exactly the same procedure; the only change is that now the algebraic equations will be a linear system.

Example

Find the general solution to the first-order linear system of ODEs

$$x' = y + \sin(t)$$
$$y' = x + \cos(t)$$

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Solution

To simplify notation, let's create two aliases

```
> alias( X(s)=laplace(x(t),t,s), Y(s)=laplace(y(t),t,s) ):
>
```

The system and its Laplace transform are

```
> SYS := { diff( x(t), t ) = y(t) + sin(t), diff( y(t), t ) = x(t) + cos(t) };
```

$$SYS := \{\frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t) + \sin(t), \frac{\partial}{\partial t}\mathrm{y}(t) = \mathrm{x}(t) + \cos(t)\}$$

```
> SYS_lap := laplace( SYS, t, s );
```

$$SYS\_lap := \{ s\,X(s) - x(0) = Y(s) + \frac{1}{s^2+1},\ s\,Y(s) - y(0) = X(s) + \frac{s}{s^2+1} \}$$

```
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```

The solution to this (linear) system of two equations is found to be

```
> SOLN_lap := solve( SYS_lap, { X(s), Y(s) } );
```

$$SOLN\_lap := \{ X(s) = \frac{x(0)\,s^3 + y(0)\,s^2 + 2\,s + x(0)\,s + y(0)}{s^4 - 1},\ Y(s) = \frac{s\,y(0) + 1 + x(0)}{s^2 - 1} \}$$

```
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```

The inverse Laplace transforms should become obvious -- or, at least more apparent -- from the partial fraction decompositions.

```
> SOLN_lap2 := convert( SOLN_lap, parfrac, s );
```

$$SOLN\_lap2 := \{ X(s) = \frac{1}{2}\frac{x(0) + 1 + y(0)}{s-1} + \frac{1}{2}\frac{x(0) + 1 - y(0)}{s+1} - \frac{s}{s^2+1},\ Y(s) = \frac{1}{2}\frac{x(0) + 1 + y(0)}{s-1} - \frac{1}{2}\frac{x(0) + 1 - y(0)}{s+1} \}$$

```
> SOLN_lap2 := collect( SOLN_lap2, { x(0), y(0) } );
```

$$SOLN\_lap2 := \{ Y(s) = \left(\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right) y(0) + \left(\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right) x(0) + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1},$$
$$X(s) = \left(\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right) y(0) + \left(\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right) x(0) + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{s}{s^2+1} \}$$

```
>
```

Thus, the general solution to this system is

```
> SOLN := invlaplace( SOLN_lap2, s, t );
```

$$SOLN := \{ x(t) = y(0)\left(\frac{1}{2}\mathbf{e}^t - \frac{1}{2}\mathbf{e}^{(-t)}\right) + x(0)\left(\frac{1}{2}\mathbf{e}^t + \frac{1}{2}\mathbf{e}^{(-t)}\right) + \frac{1}{2}\mathbf{e}^t + \frac{1}{2}\mathbf{e}^{(-t)} - \cos(t),$$
$$y(t) = y(0)\left(\frac{1}{2}\mathbf{e}^t + \frac{1}{2}\mathbf{e}^{(-t)}\right) + x(0)\left(\frac{1}{2}\mathbf{e}^t - \frac{1}{2}\mathbf{e}^{(-t)}\right) + \frac{1}{2}\mathbf{e}^t - \frac{1}{2}\mathbf{e}^{(-t)} \}$$

Points to Ponder

- Identify a basis of solutions to the homogeneous system and a particular solution.
- What other methods could be used to solve this system?

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For more information, please contact the author at one of following addresses:
Department of Mathematics, Univ. of South Carolina, Columbia, SC 29208'
(803) 777-6183
meade@math.sc.edu


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