```
>
```

Solution to Problem 11 (p. 290)

prepared by
Douglas B. Meade
(meade@math.sc.edu)

19 September 1997

Preliminaries

Start by clearing Maple's memory and then loading the linear algebra (and vector calculus) commands.

```
> restart;
> with( linalg );
Warning, new definition for norm
Warning, new definition for trace
>
```

a)

```
>
```

The path of interest in this problem can be represented by

```
> c := [ cos(t^2), sin(t^2), 0 ];
```

$$c := [\cos(t^2), \sin(t^2), 0]$$

The corresponding velocity vector and speed are

```
> v := diff( c, t );
```

$$v := [-2\sin(t^2)\,t, 2\cos(t^2)\,t, 0]$$

```
> s := norm( v, 2 );
```

$$s := 2\sqrt{|\sin(t^2)\,t|^2 + |\cos(t^2)\,t|^2}$$

```
>
```

Why doesn't Maple expand and simplify the square root? Well, it's a difficult problem for the computer to know when it is appropriate to expand an expression and when to leave it in another form. (Think about how you would define what "simple" means for a mathematical expression.) The generally accepted solution is to require the user to specifically tell the system when to expand an expression. The main Maple commands that are used for this type of expression manipulation are simplify, expand, factor, and normal. (Click on any green word to see the associated Maple help page - TRY IT!)

```
>
> simplify( s );
```

$$2\sqrt{|\sin(t^2)\,t|^2 + |\cos(t^2)\,t|^2}$$

Nothing happened! Why?

The answer is that, unless specified otherwise, Maple assumes all quantities are complex-valued. In this case, this prevents the simplification of sqrt(alpha^2) with alpha. Once again, this is not a random decision on the part of the developers. Much of the power of Maple comes from its ability to work in the complex field. The price we pay is that we have be willing to tell Maple additional information about a variable; the assume command is used for this.

```
> assume( t>0 );
```

```
> about( t );
> s := simplify( s );
```

$$s := 2\, t\sim$$

```
>
```

Additional Technical Notes

b) and c)

As discussed in class, it is possible to solve this part of the problem using geometry and trigonometry. However, if the path is not a circle, this approach will not work. The vector algebra approach is more general. Here is how that can be implemented.

IDEA: find the time, t, when the tangent vector to the path is parallel to, and points in the same direction as, the vector from the point on the path to the target point.

```
>
> target := [ 2, 0, 0 ];
```

$$target := [2, 0, 0]$$

```
> G := target - c;
```

$$G := [-\cos(t\sim^2) + 2, -\sin(t\sim^2), 0]$$

```
>
```

The cross product of the tangent vector and the direction brom the curve to the target point is

```
> cp := crossprod( v, G );
```

$$cp := [0, 0, 2 \sin(t\sim^2)^2\, t\sim - 2 \cos(t\sim^2)\, t\sim\, (-\cos(t\sim^2) + 2)]$$

This vector vanishes when th e third component is zero. Let's focus on that term for a while:

```
> eqn := cp[3];
```

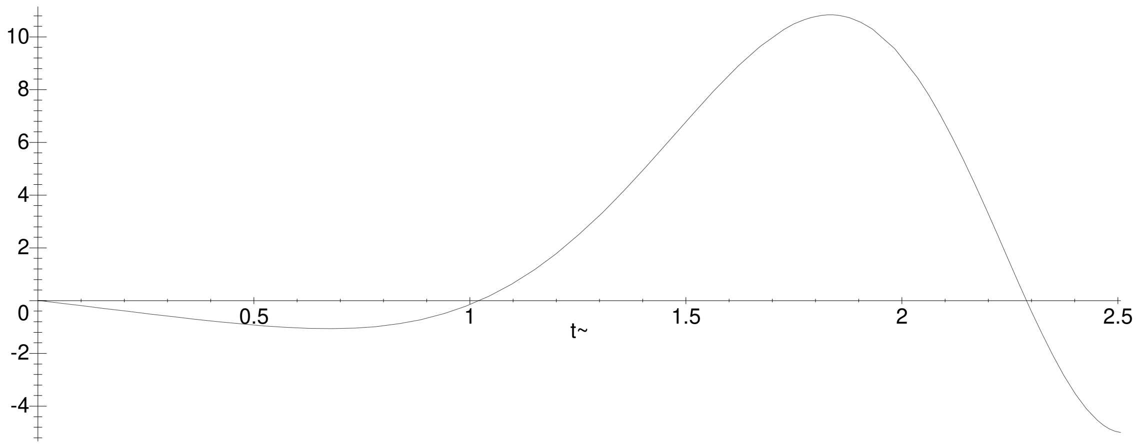
$$eqn := 2 \sin(t\sim^2)^2\, t\sim - 2 \cos(t\sim^2)\, t\sim\, (-\cos(t\sim^2) + 2)$$

```
> eqn := simplify( eqn );
```

$$eqn := 2\, t\sim - 4 \cos(t\sim^2)\, t\sim$$

We could simply try to solve this equation (it's not hard once it's simplified). Instead, let's take a look at the graph of the last element of the cross product during one revolution of the motion.

```
> plot( eqn, t=0..sqrt(2*Pi) );
```




```
>
```

If we ask Maple to solve this equation, we find ...

```
> soln := solve( eqn=0, t );
```

$$soln := 0, \frac{1}{3}\sqrt{3}\sqrt{\pi}, -\frac{1}{3}\sqrt{3}\sqrt{\pi}$$

... three solutions. Which, if any, of these is the one we want? When t=0 the cross product is zero because the velocity vector is the zero vector:

```
> subs( t=0, v );
```

$$[0, 0, 0]$$

```
>
```

The second solution is somewhat more promising:

```
> simplify( subs( t=soln[2], v ) );
```

$$\left[-\sqrt{\pi}, \frac{1}{3}\sqrt{3}\sqrt{\pi}, 0\right]$$

```
> simplify( subs( t=soln[2], G ) );
```

$$\left[\frac{3}{2}, -\frac{1}{2}\sqrt{3}, 0\right]$$

Notice that, with $t = \sqrt{\frac{\pi}{3}}$, $v = -\frac{2}{3\sqrt{\pi}}\ G$. Thus, v and G are parallel but point in *opposite* directions. This is not the time that we seek.

```
>
```

The third solution looks even less promising - we know that t>0. However, what if we write this as an angle in the fouth quadrant? A closer look at eq shows that the condition for parallel vectors simplifies to $\cos(t^2) = \frac{1}{2}$ for $t \neq 0$. A solution in the fourth quadrant is t^2 = 2*Pi - Pi/3 = 5*Pi/3, or t = sqrt(5*Pi/3).

```
> simplify( subs( t=sqrt(5*Pi/3), v ) );
```

$$\left[\sqrt{5}\sqrt{\pi}, \frac{1}{3}\sqrt{15}\sqrt{\pi}, 0\right]$$

```
> simplify( subs( t=sqrt(5*Pi/3), G ) );
```

$$\left[\frac{3}{2}, \frac{1}{2}\sqrt{3}, 0\right]$$

Now $v = \frac{2}{3\sqrt{5}\sqrt{\pi}}$ G. Since this coefficient is positive, the vectors point in the same direction and we have a solution. The point from which the particle should be released is

```
> simplify( subs( t=sqrt(5*Pi/3), c ) );
```

$$\left[\frac{1}{2}, -\frac{1}{2}\sqrt{3}, 0\right]$$

Note that this point is, as expected, in the fourth quadrant. Moreover, we have already answered part c); the particle should be released

at $t = \sqrt{\frac{5\pi}{3}}$.

```
> tREL := sqrt( 5*Pi/3 );
```

$$tREL := \frac{1}{3}\sqrt{15}\sqrt{\pi}$$

```
>
```

d)

At the time of release, the velocity is

```
> vREL := simplify( subs( t=tREL, v ) );
```

$$vREL := \left[\sqrt{5}\sqrt{\pi}, \frac{1}{3}\sqrt{15}\sqrt{\pi}, 0\right]$$

and the speed is

```
> sREL := simplify( subs( t=tREL, s ) );
```

$$sREL := \frac{2}{3}\sqrt{15}\sqrt{\pi}$$

Converting the exact speed to a floating point approximation yields:

```
> evalf( sREL );
```

$$4.576456164$$

```
>
```

e)

Assuming the particle moves at a constant velocity after release, the time from release to impact is simply the distance from the point of release to the target divided by the speed. The distance from the point of release to the target is:

```
> dist := norm( subs( t=tREL, G ), 2 );
```

$$dist := \sqrt{3}$$

```
> tHIT := dist/sREL;
```

$$tHIT := \frac{1}{10}\frac{\sqrt{3}\sqrt{15}}{\sqrt{\pi}}$$

```
> evalf( tHIT );
```

$$.3784698784$$

```
>
```

The total time from the launch to impact is

```
> tTOTAL := tREL + tHIT;
```

$$tTOTAL := \frac{1}{3}\sqrt{15}\sqrt{\pi} + \frac{1}{10}\frac{\sqrt{3}\sqrt{15}}{\sqrt{\pi}}$$

```
> evalf( tTOTAL );
```

$$2.666697960$$

Follow-Up Questions

- Suppose the particle has to be travelling faster than some threshold speed before in can be released. Then the particle may have to make several revolutions before being released. What are the times of release that will send the particle to its target at (2,0,0) during the second and third times around the path?
- What if the path is the ellipse c(t) = < t*cos(t), t/2*sin(t), 0 >?

```
>
>
```