MATH 544 (Section 501)
Prof. Meade

Exam 1
2 April 2010

University of South Carolina
Spring 2010

Name: Key
SS # (last 4 digits): ______

Instructions:

1. There are a total of 7 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. You must **show all of your work** to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	18	
2	10	
3	18	
4	12	
5	12	
6	18	
7	12	
Total	100	

Think Spring (and Summer)!

1. (18 points) Suppose the matrix A has been reduced to echelon form as shown below:

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Construct an LU factorization of A. (Display L and U.)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis for the column space of A.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -7 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 9 \\ -1 \\ 4 \end{bmatrix} \right\}$$

* columns 1, 2, & 4 of the original matrix A.

(c) Find a basis for the null space of A.

$$\begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{②} + \frac{1}{5}\text{③} \to \text{②}]{\text{①} - \frac{3}{5}\text{③} \to \text{①}} \begin{bmatrix} 2 & -4 & -2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{①} + \frac{4}{3}\text{②} \to \text{①}} \begin{bmatrix} 2 & 0 & -2/3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3\, x_3 \\ -1/3\, x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix}$$

A basis for Nul A is $\left\{ \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

2. (10 points) Find the 3×3 matrix that produces the composite 2D transformation, using homogeneous coordinates, that rotates points clockwise by $45°$ about the point $(4, -5)$.

Note: I wrote clockwise, but meant counter clockwise.

This transformation is done in 3 steps:

1. Translate by $(-4, 5)$ to move the point $(4, -5)$ to $(0, 0)$.

$$T_1 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Rotate by $\theta = -\pi/4$ (about the origin):

$$T_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Translate by $(4, -5)$ to move the origin back to $(+4, -5)$.

$$T_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

The composite transformation is given by

$$T = T_3 T_2 T_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 + \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{9\sqrt{2}}{2} - 5 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (18 points) Given $\mathbf{u}$ in R^n with $\mathbf{u}^T\mathbf{u} = 1$, let $P = \mathbf{u}\mathbf{u}^T$ and $Q = I - 2P$. Show that

(a) Show that $P^2 = P$

Note: $\vec{u}$ is $n\times 1$ (vector)
$\vec{u}\vec{u}^T$ is 1×1 (scalar)
$\vec{u}^T\vec{u}$ is $n\times n$ (matrix).

$$\begin{aligned} P^2 &= (\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T) \\ &= \vec{u}(\vec{u}^T\vec{u})\vec{u}^T \\ &= \vec{u}(1)\vec{u}^T \\ &= \vec{u}\vec{u}^T = P \end{aligned}$$

(b) Show that $P^T = P$

Using $(AB)^T = B^T A^T$.
and $(A^T)^T = A$.

$$\begin{aligned} P^T &= (\vec{u}\vec{u}^T)^T \\ &= (\vec{u}^T)^T \vec{u}^T \\ &= \vec{u}\vec{u}^T = P \end{aligned}$$

(c) Show that $Q^2 = I$

$$\begin{aligned} Q^2 &= (I-2P)^2 \\ &= (I-2P)(I-2P) \\ &= I^2 - 2P - 2P + (2P)^2 \\ &= I - 4P + 4P^2 \\ &= I - 4P + 4P = I \end{aligned}$$

because $P^2 = P$ by (a).

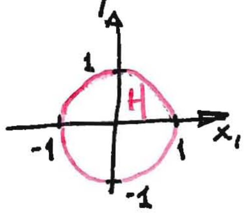
(d) Use the previous results to (i) explain why Q is invertible and (ii) find Q^{-1}.

Recall the Invertible Matrix Theorem states that when there is a matrix C with the property that $AC = I$ then $C = A^{-1}$.

Here, because $Q^2 = I$ we know that $QQ = I$ and so $Q^{-1} = Q$.

4. (12 points)

(a) Let H be the set of points inside and on the unit circle in the xy-plane. Explain why H is not a subspace of $\mathbb{R}^2$.



While $\vec{0} \in H$ $(0^2+0^2=0 \le 1)$, H is not closed under addition or scalar multiplication. For example:
$\vec{u}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\end{bmatrix}$ are both in H but $\vec{u}+\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$ is not in H.
And, if $c=2$ then $c\vec{u}=\begin{bmatrix}2\\0\end{bmatrix}$ is not in H.

(b) Let S be the set of 2×2 symmetric matrices. Recall that a matrix A is symmetric when $A^T = A$. Determine if S is a subspace of $M_{2\times 2}$.

Note: The general 2x2 symmetric matrix has the form $A=\begin{bmatrix}a & b\\ b & c\end{bmatrix}$

$S=\{A : A \in M_{2\times2},\ A^T=A\}$.

$0 \in S$ because $0^T=0$.
if $A,B \in S$ then $(A+B)^T = A^T+B^T = A+B$ so $A+B \in S$.
if $A\in S$ and $c \in \mathbb{R}$, then $(cA)^T = cA^T = cA$ so $cA \in S$. $\therefore$ S is a subspace of $M_{2\times2}$.

5. (12 points) Be sure to provide a short explanation for each answer.

(a) A 5×4 matrix A has 3 pivot columns. How many vectors are in any basis of Nul A?

Since A has 4 columns and 3 of them are pivot columns there will be 1 free variable. This free variable gives 1 vector in any basis for Nul A.

(b) A is an $m\times n$ matrix and Nul $A=\{0\}$. Assuming $m>n$, is Col $A=\mathbb{R}^m$?

In order for Col $A=\mathbb{R}^m$ there must be m pivot columns in A.
But A has only at most n pivot columns and $n<m$ so A cannot have m pivot columns. Thus, Col $A \ne \mathbb{R}^m$.

Note: Nul $A=\{0\}$ means A has 0 free variables & so has exactly n pivot columns.

(c) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent polynomials in $\mathbb{R}^5$. Is the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ a basis for $\mathbb{R}^5$?

No. $\mathbb{P}_4$ (and $\mathbb{R}^5$) are 5 dimensional. A basis must have 5 elements. No set of only 4 elements can be a basis for $\mathbb{P}_4$ (or $\mathbb{R}^5$).

Note: These references to $\mathbb{R}^5$ should have been $\mathbb{P}_4$. This is not a big deal because $\mathbb{P}_4$ is isomorphic to $\mathbb{R}^5$. I apologize for this oversight.

6. (18 points)

(a) Find the determinant of $A = \begin{bmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{bmatrix}$.

$$\det A = (-1)\det\begin{bmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{bmatrix} = (-1)(-2)\det\begin{bmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 7 \end{bmatrix}$$

$$= (-1)(-2)(3)\det\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= (-1)(-2)(3)(28-30)$$
$$= (-1)(-2)(3)(-2)$$
$$= -12$$

(b) <u>Use row operations</u> to show that $\det \begin{bmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{bmatrix} = 0$.

$$\det\begin{bmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{bmatrix} \underset{③-①\to③}{\overset{②-①\to②}{=\!=}} \det\begin{bmatrix} a & b & c \\ x & x & x \\ y & y & y \end{bmatrix}$$

$$\underset{\frac{1}{y}③\to③}{\overset{\frac{1}{x}②\to②}{=\!=}} xy \det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\overset{③-②\to③}{=\!=} xy \det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= xy \cdot 0$$ (because the final matrix has a row of all zeroes.)

7. (12 points) Use coordinate vectors to determine if each set of polynomials (i) is linearly independent, (ii) spans $\mathbf{P}_2$, and (iii) is a basis for $\mathbf{P}_2$.

(a) $\{1 - t + t^2, 1 - 2t - t^2, 1 + t^2\}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[③ - ① \to ③]{② + ① \to ②} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{③ - 2② \to ③} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

(ii) Because there is a pivot in every row, the columns of A span $\mathbb{R}^3$ (and hence $\mathbb{P}_2$).

(i) Because there is a pivot in every column, the columns of A are linearly independent. (so are the 3 polynomials)

(iii) This set of 3 polynomials in $\mathbb{P}_2$ is a basis for $\mathbb{P}_2$.

(b) $\{1 - t + t^2, 1 - 2t - t^2, 2 - 3t\}$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & -3 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[③ - ① \to ③]{② + ① \to ②} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{③ - 2② \to ③} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(i) Because there is not a pivot in every column, the columns of A are not linearly independent. (and the 3 polynomials are also linearly dependent).

(ii) Because there is not a pivot in every row, the columns of A do not span $\mathbb{R}^3$. So, the 3 polynomials do not span $\mathbb{P}_2$.

(iii) This set of 3 vectors is not a basis for $\mathbb{P}_2$.