HW Sol'n for §4.5.2

#2. $\nabla^2 u = 0 \qquad 0<x<1,\ 0<y<\pi$

$$\left.\begin{aligned} u_x(0,y) &= y - \pi/2 \\ u_x(1,y) &= \cos(y) \end{aligned}\right\} \quad 0<y<\pi.$$

$u_y(x,0) = u_x(x,\pi) = 0 \qquad 0<x<1$

We solve this problem by solving two simpler problems, and adding their solutions.

Problem 1: $\nabla^2 u = 0 \qquad 0<x<1,\ 0<y<\pi$

$$\left.\begin{aligned} u_x(0,y) &= y - \pi/2 \\ u_x(1,y) &= 0 \end{aligned}\right\} \quad 0<y<\pi$$

$u_y(x,0) = u_x(x,\pi) = 0 \qquad 0<x<1$

Separation of variables leads to considering $u = X(x)Y(y)$ where

$Y'' + \lambda Y = 0$ and $X'' - \lambda X = 0$

$Y'(0) = Y'(\pi) = 0 \qquad X'(1) = 0.$

don't forget this one!

The Y equations have non-trivial solutions for $\lambda_n = n^2 \quad (n = 0, 1, 2, \dots)$

so $Y_n(y) = \cos(ny) \quad (n = 0,1,2,\dots)$.

The corresponding solutions for X are $X_n(x) = e^{nx} + e^{2n-nx} = e^{nx} + e^{n(2-x)}$.

(for $n=0$, $X_0 = 1$ - a constant).

The solution to the full BVP is of the form:

$$u_1(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(e^{nx} + e^{n(2-x)}\right)\cos(ny).$$

To satisfy the last BC we compute:

$$u_{1,x}(x,y) = \sum_{n=1}^{\infty} a_n\left(ne^{nx} - ne^{n(2-x)}\right)\cos(ny)$$

so
$$u_{1,x}(0,y) = \sum_{n=1}^{\infty} a_n\left(n - ne^{2n}\right)\cos(ny) = \sum_{n=1}^{\infty} n(1-e^{2n})a_n\cos(ny) = y - \pi/2.$$

Thus,
$$n(1-e^{2n})a_n = \frac{2}{\pi}\int_0^{\pi}\left(y - \frac{\pi}{2}\right)\cos(ny)\,dy = \frac{2}{\pi}\,\frac{-1+(-1)^n}{n^2} = \begin{cases} 0 & n \text{ even} \\ \dfrac{-4}{\pi n^2} & n \text{ odd.} \end{cases}$$

so $a_n = -\frac{4}{\pi n^2} \frac{1}{n(1-e^{2n})} = \frac{-4}{\pi n^3(1-e^{2n})}$ for all odd integers n.

The solution is

$$u_1(x,y) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_n \left(e^{nx} + e^{n(2-x)}\right) \cos(ny) \qquad \text{(let } n = 2m-1\text{)}$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_{2m-1} \left(e^{(2m-1)x} + e^{(2m-1)(2-x)}\right) \cos((2m-1)y)$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} \frac{-4}{\pi(2m-1)^3(1-e^{2(2m-1)})} \left(e^{(2m-1)x} + e^{(2m-1)(2-x)}\right) \cos((2m-1)y).$$

Note that we have no constraint on the constant term — for any value of a_0 this function satisfies this problem.

Problem 2: $\nabla^2 u = 0 \qquad 0<x<1,\ 0<y<\pi$

$$\left.\begin{aligned} &u_x(0,y) = 0 \\ &u_x(1,y) = \cos(y) \end{aligned}\right\} 0<y<\pi.$$

$u_y(x,0) = u_y(x,\pi) = 0 \qquad 0<x<1$

This problem's solution starts out in the same way as the previous one.

For $\lambda = n^2$ we have $Y_n(y) = \cos(ny)$.

But now the X solutions must satisfy $X'' - n^2 X = 0$ with $X'(0) = 0$.

These solutions will be $X_n(x) = e^{nx} + e^{-nx} \quad (n = 0, 1, 2, \ldots)$.

Then $u_2(x,y) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \left(e^{nx} + e^{-nx}\right) \cos(ny)$.

so $u_{2,x}(x,y) = \sum_{n=1}^{\infty} b_n \left(ne^{nx} - ne^{-nx}\right) \cos(ny)$

and $u_{2,x}(1,y) = \sum_{n=1}^{\infty} b_n\, n \left(e^{n} - e^{-n}\right) \cos(ny) = \cos y$

Thus $b_1 (1)(e^1 - e^{-1}) = 1$ and $b_n n (e^n - e^{-n}) = 0$ for $n = 2, 3, \ldots$

This gives us $u_2(x,y) = \frac{b_0}{2} + \frac{1}{e^1 - e^{-1}} \left(e^{x} + e^{-x}\right) \cos(y)$ (where, again, b_0 is unspecified.)

The final solution to the full BVP is:

$$\boxed{u(x,y) = u_1(x,y) + u_2(x,y) = A + \frac{e^x + e^{-x}}{e^1 - e^{-1}} \cos y + \sum_{m=1}^{\infty} \frac{-4}{\pi(2m-1)^3} \frac{e^{(2m-1)x} + e^{(2m-1)(2-x)}}{1 - e^{2(2m-1)}} \cos((2m-1)y)}$$

for any constant A