HW Soln for §4.3

#10. $\nabla^2 u = 0 \qquad 1 < r < 2, \quad -\pi \le \theta \le \pi.$

$u(1,\theta) = \sin\theta$

$u(2,\theta) = \cos\theta$

The bounded harmonic functions on the annulus (washer) are

$1, \ \ln r, \ r^n\cos(n\theta), \ r^n\sin(n\theta), \ r^{-n}\cos(n\theta), \ r^{-n}\sin(n\theta).$

The general solution to $\nabla^2 u = 0$ for $1 < r < 2$, $-\pi \le \theta \le \pi$ is:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta) + \sum_{n=1}^{\infty} c_n r^{-n}\cos(n\theta) + \sum_{n=1}^{\infty} d_n r^{-n}\sin(n\theta) + c_0 \ln r$$

To satisfy $u(1,\theta) = \sin\theta$ we need:

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n + c_n)\cos(n\theta) + \sum_{n=1}^{\infty}(b_n + d_n)\sin(n\theta) + c_0 \cancelto{0}{\ln(1)} = \sin(\theta)$$

By inspection this will be satisfied by:

$$\frac{a_0}{2} = 0, \quad a_n + c_n = 0 \ (n \ge 1), \quad b_1 + d_1 = 1, \quad b_n + d_n = 0 \ (n \ge 2)$$

Also, $u(2,\theta) = \cos\theta$ requires:

$$u(2,\theta) = \frac{a_0}{2} + c_0\ln(2) + \sum_{n=1}^{\infty}(a_n 2^n + c_n 2^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(b_n 2^n + d_n 2^{-n})\sin(n\theta) = \cos\theta$$

which requires: $\frac{a_0}{2} + c_0\ln(2) = 0, \quad a_1 \cdot 2 + c_1 2^{-1} = 1, \quad a_n 2^n + c_n 2^{-n} = 0 \ (n \ge 2), \quad b_n 2^n + d_n 2^{-n} = 0 \ (n \ge 1)$

First, $a_0 = 0$ and then $c_0 = 0$.

Next, for $n \ge 2$: $\left.\begin{array}{l} a_n + c_n = 0 \\ 2^n a_n + 2^{-n} c_n = 0 \end{array}\right\} \Rightarrow a_n = c_n = 0 \ (n \ge 2)$ and $\left.\begin{array}{l} b_n + d_n = 0 \\ 2^n b_n + 2^{-n} d_n = 0 \end{array}\right\} \Rightarrow b_n = d_n = 0 \ (n \ge 2).$

For $n = 1$:

$$\begin{array}{r} 2(b_1 + d_1 = 1) \\ -\ 2b_1 + \frac{1}{2}d_1 = 0 \\ \hline \frac{3}{2}d_1 = 2 \end{array} \Rightarrow \begin{cases} d_1 = \frac{4}{3} \\ b_1 = -\frac{1}{4}d_1 = -\frac{1}{3} \end{cases}$$

$$\begin{array}{r} 2(a_1 + c_1 = 0) \\ -\ 2a_1 + \frac{1}{2}c_1 = 1 \\ \hline \frac{3}{2}c_1 = -1 \end{array} \Rightarrow \begin{cases} c_1 = -\frac{2}{3} \\ a_1 = \frac{2}{3} \end{cases}$$

Therefore, the full solution is:

$$\boxed{\begin{aligned} u(r,\theta) &= a_1 r\cos\theta + b_1 r\sin\theta + c_1 r^{-1}\cos\theta + d_1 r^{-1}\sin\theta \\ &= \left(\frac{2}{3}r - \frac{2}{3r}\right)\cos\theta + \left(-\frac{r}{3} + \frac{4}{3r}\right)\sin\theta \end{aligned}}$$