HW Soln for §4.2

#1. $\nabla^2 u = 0$ $0<x<1,\ 0<y<\pi$.

$u(0,y)=u(1,y)=0$ $0<y<\pi$

$u(x,0)=\sin(\pi x),\ u(x,\pi)=0$ $0<x<1$

$u=X(x)Y(y)$: $\nabla^2 u = u_{xx}+u_{yy} = X''Y + XY'' = 0$ so $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda$.

$X''-\lambda X=0$ with $X(0)=X(1)=0$

& $Y''+\lambda Y=0$ with $Y(\pi)=0$

From the X problem we know the eigenvalues are $\lambda=-(n\pi)^2$

and the corresponding eigenfunctions are $X_n(x)=\sin(n\pi x)$.

Moving to the Y equation: $Y''-(n\pi)^2 Y=0$ has general solution

$$Y_n(y) = a e^{n\pi y} + b e^{-n\pi y}$$

To have $Y(1)=0$ requires $a e^{n\pi} + b e^{-n\pi}=0$, so $b=-ae^{2n\pi}$.

This gives us $Y_n(y) = e^{n\pi y} - e^{2n\pi} e^{-n\pi y} = e^{n\pi y} - e^{2n\pi - n\pi y}$.

Putting these together gives $u_n(x,y) = \sin(n\pi x)\left(e^{n\pi y} - e^{2n\pi - n\pi y}\right)$

so the general solution to our full PDE is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\left(e^{n\pi y} - e^{2n\pi - n\pi y}\right).$$

To satisfy the final BC requires $u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\left(1-e^{2n\pi}\right) = \sin(\pi x)$

Thus $a_1\left(1-e^{2\pi}\right)=1$ and $a_n\left(1-e^{2n\pi}\right)=0$ for each $n>1$:

$$a_1 = \frac{1}{1-e^{2\pi}} \quad \text{and } a_n = 0 \ (\text{for } n>1).$$

$$\therefore \boxed{u(x,y) = \sin(\pi x)\,\frac{e^{\pi y} - e^{2\pi - \pi y}}{1-e^{2\pi}}}$$

#6. $\nabla^2 u = 0 \qquad 0<x<L,\ 0<y<K$

$u(x,0) = u_y(x,K) = 0 \quad 0<x<L$

$u(0,y) = 0,\ u(L,y) = g(y) \quad 0<y<K$

Separation of variables leads to $\frac{X''}{X} + \frac{Y''}{Y} = 0$, so $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

with $Y(0) = Y'(K) = 0$ and $X(0) = 0$.

Starting with the Y equation (because there are 2 BC for Y):

$Y'' + \lambda Y = 0,\ Y(0) = Y'(K) = 0$

The only nontrivial solutions are $\quad Y(y) = c\cos(\sigma y) + d\sin(\sigma y)$

$\qquad Y'(y) = -c\sigma\sin(\sigma y) + d\sigma\cos(\sigma y)$

Then $Y(0) = 0 \Rightarrow c = 0$.

and $Y'(K) = 0 \Rightarrow d\sigma \underbrace{\cos(\sigma K) = 0}$, $\rightarrow \sigma K = (2n-1)\frac{\pi}{2}$ ← odd multiple of $\pi/2$.

so $\sigma = (2n-1)\frac{\pi}{2K}$, $\lambda_n = +\sigma^2 = \left((2n-1)\frac{\pi}{2K}\right)^2$, $Y_n(y) = \sin\left(\frac{(2n-1)\pi y}{2K}\right)$.

Then, for X: $X'' - \lambda X = 0$ with $\lambda = \left((2n-1)\frac{\pi}{2K}\right)^2$.

so $X(x) = a e^{(2n-1)\frac{\pi}{2K}x} + b e^{-(2n-1)\frac{\pi}{2K}x}$

To satisfy $X(0) = 0$ requires: $a + b = 0$ so $b = -a$ and

$$X_n(y) = e^{(2n-1)\frac{\pi}{2K}x} - e^{-(2n-1)\frac{\pi}{2K}x}$$

The ~~general soln to the PDE is:~~

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} a_n \left(e^{(2n-1)\frac{\pi}{2K}x} - e^{-(2n-1)\frac{\pi}{2K}x}\right)\sin\left(\frac{(2n-1)\pi y}{2K}\right)}$$

To satisfy the final BC: $u(L,y) = g(y)$ requires:

$$u(L,y) = \sum_{n=1}^{\infty} a_n\left(e^{(2n-1)\frac{\pi L}{2K}} - e^{-(2n-1)\frac{\pi L}{2K}}\right)\sin\left(\frac{(2n-1)\pi y}{2K}\right) = g(y).$$

so
$$\boxed{a_n = \frac{1}{e^{(2n-1)\frac{\pi L}{2K}} - e^{-(2n-1)\frac{\pi L}{2K}}}\ \frac{2}{K}\int_0^K g(y)\sin\left(\frac{(2n-1)\pi y}{2K}\right)dy}$$