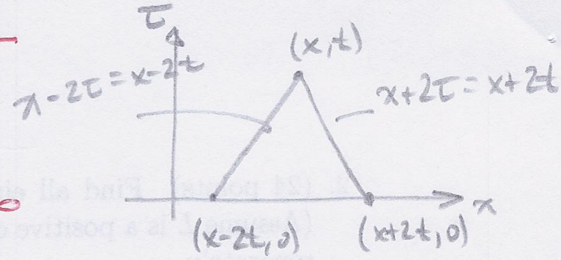
HW Sol'n for §3.2.4



#7. $u_{tt} = 4u_{xx} + t^2 \quad -\infty < x < \infty,\ t>0$

$u(x,0) = \cos(2x),\ u_t(x,0) = 1-\cos(x) \quad -\infty<x<\infty$

The d'Alembert solution for this problem is

$$u(x,t) = \frac12\left(\varphi(x-2t)+\varphi(x+2t)\right) + \frac{1}{2\cdot2}\int_{x-2t}^{x+2t} 1-\cos(s)\,ds + \frac{1}{2\cdot 2}\iint_\Delta \tau^2\,dx\,d\tau$$

$$= \frac12\left(\cos(2(x+2t))+\cos(2(x-2t))\right) + \frac14\left[s-\sin(s)\right]_{x-2t}^{x+2t}$$

$$+ \frac14\int_0^t\int_{2\tau+x-2t}^{-2\tau+x+2t}\tau^2\,dx\,d\tau$$

$$=\frac12\left(\cos(2x)\cos(4t) - \cancel{\sin(2x)\sin(4t)} + \cos(2x)\cos(4t) + \cancel{\sin(2x)\sin(4t)}\right)$$

$$+\frac14\left(\cancel{x}+2t-\sin(x+2t) - (\cancel{x}-2t)+\sin(x-2t)\right)$$

$$+\frac14\int_0^t x\tau^2\Big|_{x=2\tau+x-2t}^{x=-2\tau+x+2t} d\tau$$

$$= \cos(2x)\cos(4t) + t - \frac14\left(\sin(x+2t)-\sin(x-2t)\right)$$

$$+\frac14\int_0^t (-2\tau+\cancel{x}+2t)\tau^2 - (2\tau+\cancel{x}-2t)\tau^2\,d\tau$$

$$=\cos(2x)\cos(4t)+t-\frac14\left(\cancel{\sin x\cos(2t)} + \cos x\sin(2t) - \cancel{\sin(x)\cos(2t)} + \cos(x)\sin(2t)\right)$$

$$+\int_0^t -\tau^3 + t\tau^2\,d\tau$$

$$=\cos(2x)\cos(4t)\ \cancel{-\frac12\sin x\cos(2t)}\ + t - \frac12\cos(x)\sin(2t)$$

$$+\left(-\frac14\tau^4+\frac t3\tau^3\right)\Big|_0^t$$

$$=\cos(2x)\cos(4t)+t-\frac12\cos(x)\sin(2t)+\left(-\frac{t^4}{4}+\frac{t^4}{3}-0\right)$$

$$=\cos(2x)\cos(4t)+t-\frac12\cos(x)\sin(2t)+\frac1{12}t^4$$

The final solution is: $\boxed{u(x,t)=\cos(2x)\cos(4t)-\frac12\cos(x)\sin(2t)+t+\frac1{12}t^4}$