MATH 521 (Section 001)
Prof. Meade

University of South Carolina
Spring 2015

Exam 1
24 February 2015

Name: Key

Instructions:

1. There are a total of 4 problems on 5 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be clearly labeled and written legibly on additional sheets of paper (that I will provide). Be sure each sheet contains your name and the work for each question is clearly labeled.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

| Problem | Points | Score |
|---|---|---|
| 1 | 16 | |
| 2 | 24 | |
| 3 | 30 | |
| 4 | 30 | |
| Total | 100 | |

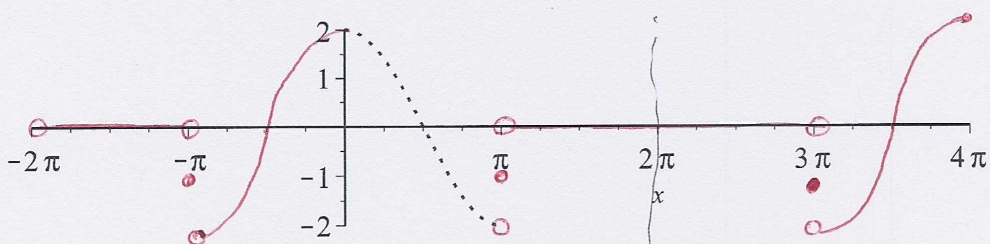
Good Luck!

1. (16 points) Let $f(x) = \begin{cases} 2\cos(x), & 0 \le x \le \pi \\ 0, & \pi < x \le 2\pi \end{cases}$.

(a) Graph the sum of the Fourier cosine series for $f(x)$ on $[-2\pi, 4\pi]$ on the axes provided.

extend to $[-2\pi, 0]$ by the even extension.

then make periodic w/ period $4\pi$



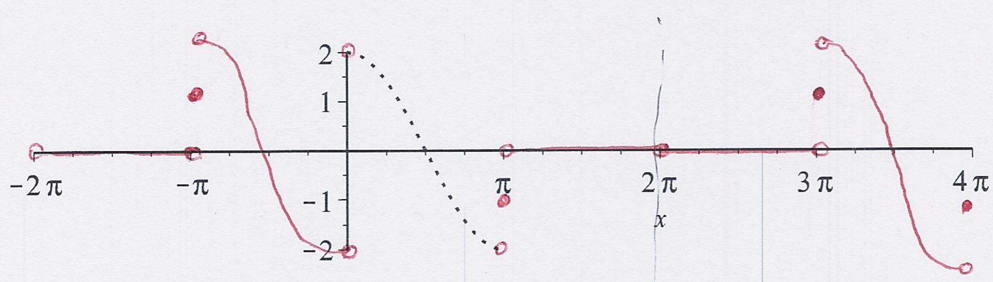

even extension

periodic extension

(b) Graph the sum of the Fourier sine series for $f(x)$ on $[-2\pi, 4\pi]$ on the axes provided.

extend to $[-2\pi, 0]$ by the odd extension

then make periodic w/ period $4\pi$




odd extension

periodic extension

2. (24 points) Find all eigenvalues and eigenvectors of $X'' + \lambda X = 0$, $X'(0) = X(L) = 0$. (Assume $L$ is a positive constant.) Be sure to consider the cases with $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ separately.

Case I: $\lambda < 0$ $(\lambda = -\sigma^2)$

$X'' - \sigma^2 X = 0$ has gen'l sol'n $X(x) = a e^{\sigma x} + b e^{-\sigma x}$

$X'(x) = \sigma a e^{\sigma x} - \sigma b e^{-\sigma x}$

$X'(0) = \sigma a - \sigma b = 0$

$= \sigma(a-b) = 0$

$a = b$ $(\sigma \neq 0)$.

$X(L) = a e^{\sigma L} + b e^{-\sigma L} = a\underbrace{(e^{\sigma L} + e^{-\sigma L})}_{>0} = 0 \Rightarrow a = 0, \; b = 0$ ∴ no nontrivial sol'n.

Case II: $\lambda = 0$

$X'' = 0$ has gen'l soln $X(x) = ax + b$

$X'(x) = a$

$X'(0) = a = 0$

$X(L) = aL + b = b = 0$ ∴ no non-trivial sol'ns.

Case III: $\lambda > 0$ $(\lambda = \sigma^2)$

$X'' + \sigma^2 X = 0$ has gen'l soln $X(x) = a\cos(\sigma x) + b\sin(\sigma x)$

$X'(x) = -a\sigma\sin(\sigma x) + b\sigma\cos(\sigma x)$

$X'(0) = b\sigma\cos(0) = b\sigma = 0 \Rightarrow b = 0.$

$X(L) = a\cos(\sigma L) = 0 \Rightarrow \cos(\sigma L) = 0$

$\sigma L = (2n-1)\frac{\pi}{2}$ (odd multiple of $\pi/2$)

$\sigma = (2n-1)\frac{\pi}{2L}$

∴ eigenvalues: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$

eigenfunctions: $X_n(x) = \cos\left(\frac{(2n-1)\pi}{2L}\right)$

3. (30 points) Suppose you have a long, thin, homogeneous bar of length $L$, with sides poorly insulated. Heat radiates freely from the bar along its length. Assuming a positive transfer coefficient $A$ and a constant temperature $T$, in the surrounding medium. Assume the ends of the bar are insulated, and the initial temperature is $f(x)$. The boundary value problem satisfied by $u = u(x,t)$ is:

$$u_t = ku_{xx} - A(u - T), \quad 0 < x < L, t > 0 \tag{1}$$
$$u_x(0,t) = u_x(L,t) = 0, \quad t > 0 \tag{2}$$
$$u(x,0) = f(x), \quad 0 < x < L. \tag{3}$$

(a) Let $w(x,t) = u(x,t) - T$. Find the boundary value problem satisfied by $w$.

$w_t = u_t$ $\quad$ $u_t = ku_{xx} - A(u-T)$ $\quad$ $u_x(0,t) = 0$ $\quad$ $u(x,0) = f(x)$.

$w_x = u_x$ $\quad$ $\underline{w_t = kw_{xx} - Aw}$ $\quad$ $\underline{w_x(0,t) = 0}$ $\quad$ $w(x,0) + T = f(x)$

$w_{xx} = u_{xx}$ $\quad$ $\underline{w(x,0) = f(x) - T}$.

$u_x(L,t) = 0$

$\underline{w_x(L,t) = 0}$

(b) Let $w(x,t) = e^{\alpha x + \beta t} v(x,t)$. Show that, with $\alpha = 0$ and $\beta = A$, $v$ satisfies the boundary value problem:

$$v_t = kv_{xx}, \quad 0 < x < L, t > 0 \tag{4}$$
$$v_x(0,t) = v_x(L,t) = 0, \quad t > 0 \tag{5}$$
$$v(x,0) = (f(x) - T), \quad 0 < x < L. \tag{6}$$

$w_t = e^{\alpha x+\beta t} v_t + \beta e^{\alpha x+\beta t} v$
$= e^{\alpha x+\beta t}(v_t + \beta v)$

$w_x = e^{\alpha x+\beta t}(v_x + \alpha v)$.

$w_{xx} = e^{\alpha x+\beta t}\left((v_x + \alpha v)_x + \alpha(v_x + \alpha v)\right)$
$= e^{\alpha x+\beta t}(v_{xx} + 2\alpha v_x + \alpha^2 v)$

$w_t = kw_{xx} - Aw$

$e^{\alpha x+\beta t}(v_t + \beta v) = k\left(e^{\alpha x+\beta t}\right)(v_{xx} + 2\alpha v_x + \alpha^2 v) - Ae^{\alpha x+\beta t} v$

$v_t + \beta v = k(v_{xx} + 2\alpha v_x + \alpha^2 v - Av)$

$v_t = kv_{xx} + 2k\alpha v_x + (\alpha^2 - A - \beta)v$

$= kv_{xx}$ if we choose $\underline{\alpha = 0}$ & $\alpha^2 - A - \beta = 0$

$\underline{\beta = A + \alpha^2 = -A}$

Then $0 = w_x(0,t) = e^{-At} v_x(0,t) \Rightarrow \underline{v_x(0,t) = 0}$

$0 = w_x(L,t) = e^{-At} v_x(L,t) \Rightarrow \underline{v_x(L,t) = 0}$

(c) Solve the boundary value problem for $v$ found in (b).

$w(x,0) = f(x) - T$

$e^{-A\cdot 0} v(x,0) = f(x) - T$

$\therefore \underline{v(x,0) = f(x) - T}$.

The problem in (4), (5), (6) is a standard heat equation with insulated ends. Its solution is

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where $a_n = \frac{2}{L}\int_0^L (f(x) - T)\cos\left(\frac{n\pi x}{L}\right) dx \quad (n = 0, 1, 2, \ldots)$

Note that the overall soln to the original problem, (1), (2), (3), is:

$u(x,t) = w(x,t) + T = T + e^{-At} v(x,t) = T + e^{-At}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)$.

4. (30 points) Consider the following boundary value problem for a wave equation:

$$u_{tt} = 9u_{xx} + 4x, \qquad 0 < x < 1, t > 0 \tag{7}$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0 \tag{8}$$
$$u(x,0) = 0, u_t(x,0) = 1, \qquad 0 < x < 1. \tag{9}$$

(a) Find a function $f(x)$ such that $u(x,t) = v(x,t) + f(x)$ where $v(x,t)$ satisfies a standard wave equation ($v_{tt} = c^2 v_{xx}$) with both ends held at the rest position ($v(0,t) = v(1,t) = 0$).

$u(x,t) = v(x,t) + f(x)$
$u_t = v_t$
$u_{tt} = v_{tt}$
$u_x = v_x + f'(x)$
$u_{xx} = v_{xx} + f''(x)$

$u_{tt} = 9u_{xx} + 4x$
$v_{tt} = 9(v_{xx} + f''(x)) + 4x$
$= 9v_{xx} + 9f''(x) + 4x$
$= 9v_{xx}$
provided $9f''(x) + 4x = 0$

$9f''(x) + 4x = 0$
$f''(x) = -\frac{4}{9}x$
$f'(x) = -\frac{2}{9}x^2 + a$
$f(x) = -\frac{2}{27}x^3 + ax + b$

$u(0,t) = v(0,t) + f(0) = 0 \Rightarrow f(0) = 0.$
$u(1,t) = v(1,t) + f(1) = 0 \Rightarrow f(1) = 0$
$f(0) = b = 0$
$f(1) = -\frac{2}{27} + a = 0 \Rightarrow a = +\frac{2}{27}$
$\therefore f(x) = -\frac{2}{27}x^3 + \frac{2}{27}x$

(b) Find the initial conditions satisfied by $v(x,t)$.

$u(x,0) = v(x,0) + f(x) = 0$ $\qquad$ $u_t(x,0) = v_t(x,0) = 1$
$v(x,0) = -f(x)$

(c) Find the general form for the solution to the boundary value problem for $v$. (If you don't get a definite answer for (b), use the initial conditions $v(x,0) = F(x)$ and $v_t(x,0) = G(x)$.)

$v_{tt} = 9v_{xx}$
$v(0,t) = v(1,t) = 0$
$v(x,0) = -f(x)$
$v_t(x,0) = 1$

This is the heat equation with $c=3$, $L=1$ and ends held at zero. The general solution is

$$v(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n \cos(3n\pi t) + b_n \sin(3n\pi t)\right)$$

where $a_n = 2\int_0^1 -f(x)\sin(n\pi x)\,dx$
$3n\pi b_n = 2\int_0^1 1 \sin(n\pi x)\,dx$

(d) What is the solution $u(x,t)$ to the original boundary value problem? (Express Fourier coefficients as unevaluated definite integrals.)

$u(x,t) = v(x,t) + f(x)$

$$= \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n \cos(3n\pi t) + b_n \sin(3n\pi t)\right) - \frac{2}{27}x^3 + \frac{2}{27}x.$$