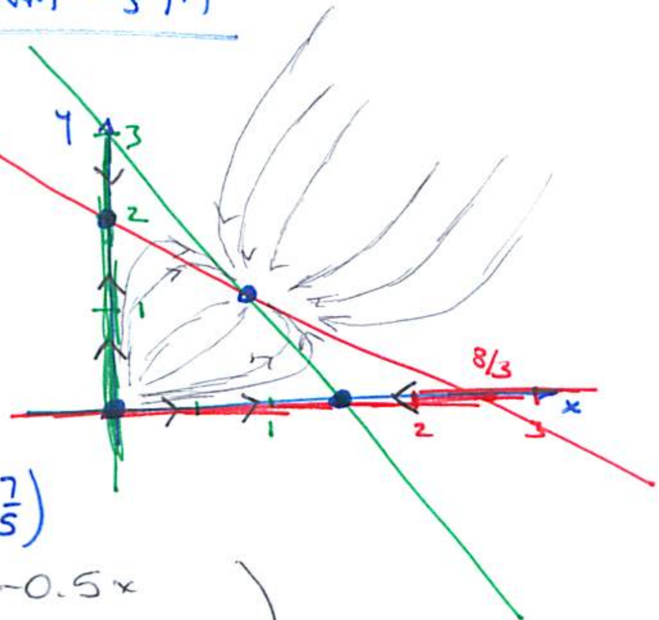
HW Solution - §9.4

#1. $\dfrac{dx}{dt} = x(1.5 - x - 0.5y)$

$\dfrac{dy}{dt} = y(2 - y - 0.75x)$

(a)



(b)

- x-nullclines: $x = 0$ & $x + \frac{1}{2}y = 1.5$
- y-nullclines: $y = 0$ & $\frac{3}{4}x + y = 2$

critical pts: $(0,0)$, $(0,2)$, $(\frac{3}{2}, 0)$, $(\frac{4}{5}, \frac{7}{5})$

(c) $J = \begin{pmatrix} 1.5 - 2x - 0.5y & -0.5x \\ -0.75y & 2 - 2y - 0.75x \end{pmatrix}$

so $J(0,0) = \begin{pmatrix} 1.5 & 0 \\ 0 & 2 \end{pmatrix} \Longrightarrow$ (0,0) is an unstable node

$J(0,2) = \begin{pmatrix} 0.5 & 0 \\ -1.5 & -2 \end{pmatrix} \Longrightarrow$ (0,2) is an unstable saddle point

$J(\frac{3}{2},0) = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{4} \\ 0 & \frac{7}{8} \end{pmatrix} \Longrightarrow$ $(\frac{3}{2}, 0)$ is an unstable saddle point

$J(\frac{4}{5},\frac{7}{5}) = \begin{pmatrix} -4/5 & -2/5 \\ -\frac{21}{20} & -7/5 \end{pmatrix}$

$p = -\frac{11}{5}$

$q = \frac{21}{25} - \frac{42}{100} = \frac{21}{50} > 0$

$\Delta = p^2 - 4q = \frac{121}{25} - \frac{42}{25} = \frac{79}{25} > 0$

$\Longrightarrow$ $(\frac{4}{5}, \frac{7}{5})$ is an asymptotically stable node.

(f) Solutions approach the critical point $(\frac{4}{5}, \frac{7}{5})$.

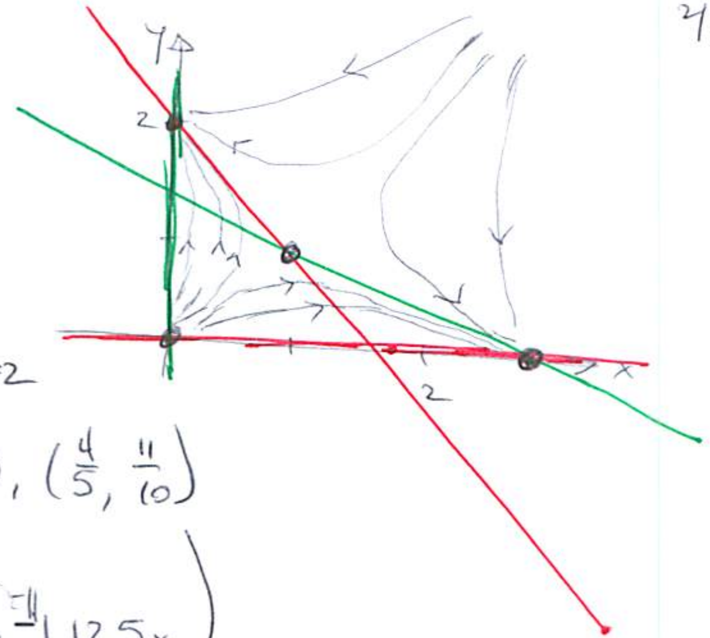
Note that $\sigma_1 = \sigma_2 = 1$, $\varepsilon_1 = 1.5$, $\varepsilon_2 = 2$, $\alpha_1 = 0.5$, and $\alpha_2 = 0.75$.

From $\sigma_1\sigma_2 = 1$ and $\alpha_1\alpha_2 = \frac{3}{8}$, where $\alpha_1\alpha_2 < \sigma_1\sigma_2$, we see that this model has weak competition, so the mutually positive critical point is expected to be ~~a positive~~ asymptotically stable node.

#3. $\frac{dx}{dt} = x(1.5 - \frac{1}{2}x - y)$

$\frac{dy}{dt} = y(2 - y - 1.125x)$

(a)



(b) x-nullclines : $x = 0$ & $\frac{1}{2}x + y = \frac{3}{2}$

y-nullclines : $y = 0$ & $1.125x + y = 2$

critical pts: $(0,0)$, $(3,0)$, $(0,2)$, $(\frac{4}{5}, \frac{11}{10})$

(c) $J = \begin{pmatrix} 1.5 - x - y & -x \\ -1.125y & 2 - 2y - 1.125x \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 1.5 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ $(0,0)$ is an unstable node

$J(3,0) = \begin{pmatrix} -1.5 & -3 \\ 0 & -1.375 \end{pmatrix} \Rightarrow$ $(3,0)$ is an asymptotically stable node

$J(0,2) = \begin{pmatrix} -0.5 & 0 \\ -2.25 & -2 \end{pmatrix} \Rightarrow$ $(0,2)$ is an asymptotically stable node

$J(\frac{4}{5}, \frac{11}{10}) = \begin{pmatrix} -0.4 & -4/5 \\ -\frac{99}{80} & -\frac{11}{10} \end{pmatrix}$ $p = -\frac{1}{2}$, $q = -\frac{11}{20}$ $\Rightarrow$ $(\frac{4}{5}, \frac{11}{10})$ is an unstable saddle point.

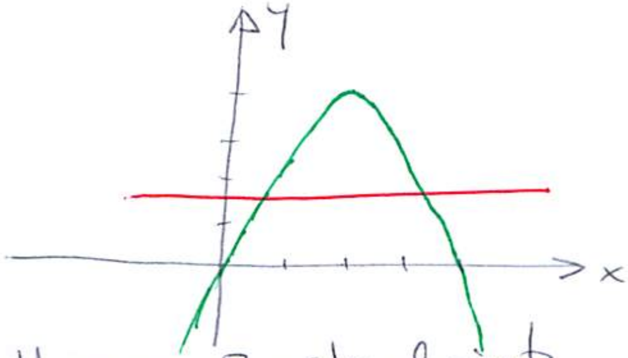
(f) As $t \to \infty$, solutions approach either $(3,0)$ or $(0,2)$, depending on where in the 1st quadrant you start.

Note: $\varepsilon_1 = 1.5$, $\sigma_1 = \frac{1}{2}$, $\alpha_1 = 1$

$\varepsilon_2 = 2$, $\sigma_2 = 1$, $\alpha_2 = +\frac{9}{8}$

$\sigma_1\sigma_2 = \frac{1}{2}$ $\alpha_1\alpha_2 = \frac{9}{8}$ so $\alpha_1\alpha_2 > \sigma_1\sigma_2$ and this is an example with strong competition (which prevents both the species from surviving in the long run)

#13. $x' = -4x + y + x^2$
$y' = \frac{3}{2}\alpha - y$



(a) — x nullcline: $y = 4x - x^2$
— y nullcline: $y = \frac{3\alpha}{2}$

For $\alpha < \frac{8}{3}$, $\frac{3\alpha}{2} < \frac{3}{2}\cdot\frac{8}{3} = 4$ and there are 2 critical points

For $\alpha = \frac{8}{3}$, $\frac{3\alpha}{2} = 4$ and there is 1 critical point, viz. $(2,4)$.

For $\alpha > \frac{8}{3}$, $\frac{3\alpha}{2} > 4$ and there are no critical points.

(b) For $\alpha < \frac{8}{3}$ the critical points have $y = \frac{3\alpha}{2} = 4x - x^2$

so $x^2 - 4x + \frac{3\alpha}{2} = 0$. By the quadratic formula:

$$x = \frac{1}{2}\left(4 \pm \sqrt{16 - 6\alpha}\right) = 2 \pm 2\sqrt{1 - \frac{3\alpha}{8}}$$

The critical points are $\left(2 + 2\sqrt{1 - \frac{3\alpha}{8}}, \frac{3\alpha}{2}\right)$ and $\left(2 - 2\sqrt{1 - \frac{3\alpha}{8}}, \frac{3\alpha}{2}\right)$.

(c) When $\alpha = 2$ the critical points are $\left(2 + 2\sqrt{1 - \frac{3}{4}}, 3\right) = (3,3)$
and $\left(2 - 2\sqrt{1 - \frac{3}{4}}, 3\right) = (1,3)$.

$$J = \begin{pmatrix} -4+2x & 1 \\ 0 & -1 \end{pmatrix}$$ so $J(3,3) = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow (3,3)$ is a saddle point

and $J(1,3) = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow (1,3)$ is an asympt. stable node.

(d) As reported above, the critical points coincide when $\alpha = \frac{8}{3}$, and the critical point is $(2,4)$.

$J(2,4) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ which has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$.

(e) Consider $\alpha = 3$ ($> 8/3$). There are no critical points.