Solution to §7.9 #3

$$\vec{x}\,' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

1. Find homog. sol'n: $\vec{x}_c$

$$\det(A-\lambda I) = (2-\lambda)(-2-\lambda)+5 = \lambda^2+1 = 0 \iff \lambda = \pm i.$$

$$\lambda = i:\ (A-iI)\vec{\xi} = \vec{0}\ :\ \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\vec{\xi} = \vec{0} \implies \vec{\xi} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\vec{\xi}e^{\lambda t} = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(\cos t + i\sin t) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right) + i\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right)$$

so $\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$ & $\vec{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$

$$\text{Then } \vec{x}_c = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} = \underbrace{\begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}}_{\mathbf{X}(t).}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

2. Find a particular solution: $\vec{x}_p = \mathbf{X}(t)\vec{u}(t)$ (where $\mathbf{X}\vec{u}\,' = \vec{g}$).

$$\vec{u}\,' = \mathbf{X}^{-1}\vec{g} \div \quad \mathbf{X}^{-1} = \frac{1}{\underbrace{(2\cos t - \sin t)\sin t - \cos t(2\sin t + \cos t)}_{= -\sin^2 t - \cos^2 t = -1.}}\begin{pmatrix} \sin t & -2\sin t - \cos t \\ -\cos t & 2\cos t - \sin t \end{pmatrix} = \begin{pmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & -2\cos t + \sin t \end{pmatrix}$$

$$\vec{x}_p{}' = \mathbf{X}^{-1}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \sin t\cos t + (2\sin t + \cos t)\sin t \\ -\cos^2 t + (-2\cos t + \sin t)\sin t \end{pmatrix} = \begin{pmatrix} 2\sin t\cos t + 2\sin^2 t \\ -2\sin t\cos t - 2\cos^2 t + 1 \end{pmatrix}$$

where we used $\sin^2 t = 1 - \cos^2 t$.

Recall: $\int 2\sin t\cos t\,dt = \sin^2 t$, $\int 2\sin^2 t\,dt = t - \sin t\cos t$, $\int 2\cos^2 t\,dt = t + \sin t\cos t$.

$$\text{Then } \vec{u} = \int\begin{pmatrix} \sin^2 t + t - \sin t\cos t \\ -\sin^2 t - (t + \sin t\cos t) + t \end{pmatrix} = \begin{pmatrix} t + \sin^2 t - \sin t\cos t \\ -\sin^2 t - \sin t\cos t \end{pmatrix}$$

and so

$$\vec{x}_p = \mathbf{X}\vec{u} = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} t + \sin^2 t - \sin t\cos t \\ -\sin^2 t - \sin t\cos t \end{pmatrix}$$

$$= \begin{pmatrix} (2\cos t - \sin t)t + (2\cos t - \sin t)(\sin^2 t - \sin t\cos t) - (2\sin t + \cos t)(\sin^2 t + \sin t\cos t) \\ \cos t\cdot t + \cos t(\sin^2 t - \sin t\cos t) - \sin t(\sin^2 t + \sin t\cos t) \end{pmatrix}$$

$$= t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} \cos t(\cancel{2\sin^2 t} + \cancel{\sin^2 t} - \cancel{2\sin^2 t} - \cancel{\sin^2 t}) + \sin t(-2\cos^2 t - \sin^2 t - 2\sin^2 t - \cos^2 t) \\ \cos t(\cancel{\sin^2 t} - \cancel{\sin^2 t}) + \sin t(-\cos^2 t - \sin^2 t) \end{pmatrix}$$

$$= t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \sin t\begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

3. General sol'n:

$$\vec{x} = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} + t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - \sin t\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$