Homework Solutions · § 4.4

#4. $y''' + y' = \sec t \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$

Homog: $r^3 + r = r(r^2+1) = 0 \Rightarrow r = 0, r = \pm i \Rightarrow y_1 = 1, y_2 = \cos t, y_3 = \sin t.$

Var. of Param: $y_p = u_1 \cdot 1 + u_2 \cdot \cos t + u_3 \cdot \sin t$

where $\begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sec t \end{bmatrix}.$ } look at these first

$$\begin{cases} -\sin t \; u_2' + \cos t \; u_3' = 0 & \} \times \sin t \\ -\cos t \; u_2' - \sin t \; u_3' = \frac{1}{\cos t} & \} \times \cos t \end{cases}$$

$(-\sin^2 t - \cos^2 t)\, u_2' = 1 \Rightarrow u_2' = -1.$ so $-\sin t\,(-1) + \cos t\; u_3' = 0$

$u_3' = -\frac{\sin t}{\cos t} = -\tan t$

Then, from the 1st equation: $u_1' = -\cos t \cdot u_2' - \sin t \cdot u_3' = \cos t + \frac{\sin^2 t}{\cos t} = \frac{\cos^2 t + \sin^2 t}{\cos t} = \sec t.$

Integrating: $u_1' = \sec t \Rightarrow u_1 = \ln(\sec t + \tan t)$

$u_2' = -1 \Rightarrow u_2 = -t$

$u_3' = -\tan t \Rightarrow u_3 = -\ln(\sec t) = \ln\left(\frac{1}{\sec t}\right) = \ln(\cos t)$

Thus a particular solution is $\underline{y_p = \ln(\sec t + \tan t) - t\cos t + \ln(\cos t) \cdot \sin t}$

General solution: $y = c_1 + c_2 \cos t + c_3 \sin t + \ln(\sec t + \tan t) - t\cos t + \ln(\cos t)\sin t.$

#9. Solve the IVP: $y''' + y' = \sec t$, $y(0) = 2$, $y'(0) = 1$, $y''(0) = -2$, $\cancel{y'''(0) = 1}$

For #4, the general solution is:

$y = c_1 + c_2 \cos t + c_3 \sin t + \ln(\sec t + \tan t) - t\cos t + \ln(\cos t) \cdot \sin t$

$y' = -c_2 \sin t + c_3 \cos t + \sec t - \cos t + t\sin t + (-\tan t)\sin t + \ln(\cos t)\cos t$

$y'' = -c_2 \cos t - c_3 \sin t + \tan^2 t + \sin t + \sin t + t\cos t + (-\sec^2 t)\sin t - \tan t \cos t + (-\tan t)\cos t - \ln(\cos t)\sin t$

Thus:

$y(0) = c_1 + c_2 + \ln(1+0) + 0 + \ln(1) \cdot 0 = c_1 + c_2 = 2 \qquad \Rightarrow c_1 = 2 - c_2 = 0$

$y'(0) = c_3 + 1 - 1 - 0 + 0 + \ln(1) \cdot 1 = c_3 + 0 = 1 \Rightarrow c_3 = 1$

$y''(0) = -c_2 + 0 + 0 + 0 + 0 + 0 - 0 + 0 - 0 = -c_2 = -2 \Rightarrow c_2 = 2$

Final solution: $\underline{y = 0 + 2\cos t + \sin t + \ln(\sec t + \tan t) - t\cos t + \ln(\cos t)\sin t}$