§2.1 Homework Solutions

#8c. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

This DE is 1st order and linear. To solve it, first put it in standard form:

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$$

The integrating factor is $\mu(t)$ where $\frac{d\mu}{dt} = \frac{4t}{1+t^2}\mu$ (separable)

$$\int \frac{d\mu}{\mu} = \int \frac{4t}{1+t^2}\,dt \qquad u = 1+t^2, \quad du = 2t\,dt$$

you can always choose $C=0$ when finding μ.

$$\ln \mu = 2\int \frac{du}{u} = 2\ln u + C = 2\ln(1+t^2)$$

$$\mu = e^{2\ln(1+t^2)} = (1+t^2)^2.$$

Multiply the DE by $\mu(t) = (1+t^2)^2$:

$$\underbrace{(1+t^2)^2 y' + 4t(1+t^2)y}_{} = (1+t^2)^{-1}$$

We choose μ so we can combine the terms into 1 derivative.

$$\left((1+t^2)^2 y\right)' = \frac{1}{1+t^2}$$

Integrate: $(1+t^2)^2 y = \arctan(1+t^2) + C$

$$y = \frac{\arctan(1+t^2)}{(1+t^2)^2} + \frac{C}{(1+t^2)^2}.$$

#16. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$, $y(\pi) = 0$ $(t>0)$.

1st order linear; $\frac{d\mu}{dt} = \frac{2}{t}\mu$: $\int \frac{d\mu}{\mu} = 2\int\frac{dt}{t}$ $\quad \ln\mu = 2\ln t = \ln(t^2)$

so $\mu = t^2$.

$$(t^2 y)' = t^2 \cdot \frac{\cos t}{t^2} = \cos t$$

Integrate: $t^2 y = \sin t + C$

Plug in IC: $\pi^2 y_0 = \sin\pi + C = C$

$$t^2 y = \sin t + \pi^2 y_0$$

$$y = \frac{\sin t}{t^2} + \frac{\pi^2 y_0}{t^2}.$$

Well, I solved this for $y(\pi) = y_0$. With $y_0 = 0$, the solution is

$$y = \frac{\sin t}{t^2}.$$

#31. $y' - \frac{3}{2}y = 3t + 2e^{t}$, $y(0) = y_0$.

1st order & linear : $\frac{d\mu}{dt} = -\frac{3}{2}\mu$ $\quad \frac{d\mu}{\mu} = -\frac{3}{2}dt$ $\quad \ln\mu = -\frac{3}{2}t$ $\quad \therefore \mu = e^{-\frac{3}{2}t}$.

$$\left(e^{-\frac{3}{2}t}y\right)' = e^{-\frac{3}{2}t}(3t+2e^{t}) = te^{-\frac{3}{2}t} + 2e^{-\frac{1}{2}t}.$$

Integrate: $e^{-\frac{3}{2}t}y = 3\underbrace{\int te^{-\frac{3}{2}t}dt} + 2\frac{1}{(-\frac{1}{2})}e^{-\frac{1}{2}t} + C$

$$\int te^{-\frac{3}{2}t}dt = t\left(-\frac{2}{3}e^{-\frac{3}{2}t}\right) - \int -\frac{2}{3}e^{-\frac{3}{2}t}dt$$

$$\begin{pmatrix} u=t & dv=e^{-\frac{3}{2}t}dt \\ du=dt & v=-\frac{2}{3}e^{-\frac{3}{2}t} \end{pmatrix} \quad \begin{aligned} &= -\frac{2}{3}te^{-\frac{3}{2}t} - \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)e^{-\frac{3}{2}t} \\ &= -\frac{2}{3}te^{-\frac{3}{2}t} - \frac{4}{9}e^{-\frac{3}{2}t} \end{aligned}$$

$$= 3\left(-\frac{2}{3}te^{-\frac{3}{2}t} - \frac{4}{9}e^{-\frac{3}{2}t}\right) - 4e^{-\frac{1}{2}t} + C.$$

Then, multiplying by $e^{\frac{3}{2}t}$:

$$y = e^{\frac{3}{2}t}\left(\frac{-2(3)}{3}te^{-\frac{3}{2}t} - \frac{4(3)}{9}e^{-\frac{3}{2}t} - 4e^{-\frac{1}{2}t} + C\right)$$

$$= -2t - \frac{4}{3} - 4e^{t} + Ce^{\frac{3}{2}t}$$

The dominant term in the solution is $Ce^{3/2t}$.

If $C > 0$ then $y \to \infty$ and if $C \le 0$ then $y \to -\infty$ (as $t \to \infty$).

To determine the sign of C, we apply the IC:

$$y_0 = -2\cdot 0 - \frac{4}{3} - 4e^{0} + Ce^{0} = -\frac{4}{3} - 4 + C = -\frac{16}{3} + C$$

so $C = y_0 + \frac{16}{3}$.

When $y_0 > \frac{-16}{3}$, $C > 0$ and so $y \to +\infty$ as $t \to \infty$.

and when $y_0 \le -\frac{16}{3}$, $C \le 0$ and so $y \to -\infty$ as $t \to \infty$.