MATH 520 (Section 001)
Prof. Meade

University of South Carolina
Spring 2008

Exam 3
April 10, 2008

Name: Key

Instructions:

1. There are a total of 6 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. *Calculators may not be used for any portion of this exam.*
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	15	
2	20	
3	10	
4	15	
5	20	
6	25	
Total	105	

Good Luck!

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have read the instructions for this exam and have neither given nor received unauthorized aid on this exam.

Signature / Date

1. (15 points) Find the general solution to the nonhomogeneous first-order linear differential equation

$$y' + 2ty = 4t^3 e^{-t^2}.$$

Homogeneous soln: $y' + 2ty = 0$ (separable)

Not Constant Coeff!

$$\frac{dy}{dt} = -2ty$$

$$\frac{dy}{y} = -2t\,dt$$

$$\ln|y| = -t^2 + C$$

$$|y| = e^{-t^2+C}$$

$$\boxed{y_c = Ce^{-t^2}}$$

Particular Soln: $y_p = v e^{-t^2}$

$$y_p' = v'e^{-t^2} - 2tve^{-t^2}$$

$$y_p' + 2ty_p = v'e^{-t^2} - 2tve^{-t^2} + 2tve^{-t^2} = v'e^{-t^2} = 4t^3e^{-t^2}$$

So $v' = 4t^3 \implies v = t^4$ & $y_p = t^4 e^{-t^2}$.

The general solution is $y = Ce^{-t^2} + t^4e^{-t^2}$.

2. (20 points) Use the Method of Variation of Parameters to find a particular solution to

$$y'' + 2y' + y = \frac{e^{-t}}{t}, \qquad t > 0$$

Homogeneous: $y'' + 2y' + y = 0$ (Const. Coeff.)

$y = e^{rt}$: $r^2 + 2r + 1 = 0$
$(r+1)^2 = 0 \Rightarrow r = -1$ (mult. 2).

$y_c = c_1 e^{-t} + c_2 t e^{-t}$. $(y_1 = e^{-t},\ y_2 = te^{-t})$.

Variation of Parameters: $y_p = v_1 e^{-t} + v_2 t e^{-t}$.

where $\begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t}/t \end{pmatrix}$

$W[y_1, y_2] = \det \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{pmatrix} = (1-t)e^{-2t} + te^{-2t}$
$= e^{-2t}$.

Then $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{e^{-2t}} \begin{pmatrix} (1-t)e^{-t} & -te^{-t} \\ e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ e^{-t}/t \end{pmatrix} = \begin{pmatrix} -1 \\ 1/t \end{pmatrix}$

so $\left.\begin{array}{l} u_1' = -1 \Rightarrow u_1 = -t \\ u_2' = 1/t \Rightarrow u_2 = \log(t) \end{array}\right\} \Rightarrow y_p = -te^{-t} + \ln(t)\, te^{-t}$

3. (10 points) Consider the autonomous differential equation $y' + 4y^2 - y^4 = 0$.

(a) Find all equilibrium solutions.

$y' = -4y^2 + y^4 = -y^2(4 - y^2) = -y^2(2-y)(2+y) = 0$

So the equilibrium solutions are $y = 0$
$y = 2$
$y = -2$.

(b) Sketch the phase line.

(c) Determine the stability of each equilibrium solution.

$y = 2$ is unstable

$y = 0$ is semi-stable

$y = -2$ is stable.

4. (15 points) Consider the system of differential equations

$$x' = 3y^2, \qquad y' = \cos(x).$$

(a) Find the differential equation for the trajectories of this system.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\cos(x) = \frac{dy}{dx} 3y^2$$

$$\frac{dy}{dx} = \frac{\cos(x)}{3y^2} \text{ (separable)}$$

(b) Find explicit formulas for the trajectories of this system.

$$\frac{dy}{dx} = \frac{\cos(x)}{3y^2}$$

$$\int 3y^2\,dy = \int \cos(x)\,dx$$

$$y^3 = \sin(x) + C$$

$$y = (\sin(x) + C)^{1/3}.$$

5. (20 points) Consider the system of differential equations

$$\begin{aligned} x' &= x - 2y + 3 \\ y' &= -2x + y + 3. \end{aligned}$$

(a) [8 points] Find, and sketch, the nullclines for this system. Identify any equilibrium solution for this system.

x-nullcline: $x - 2y + 3 = 0$

$x + 3 = 2y$

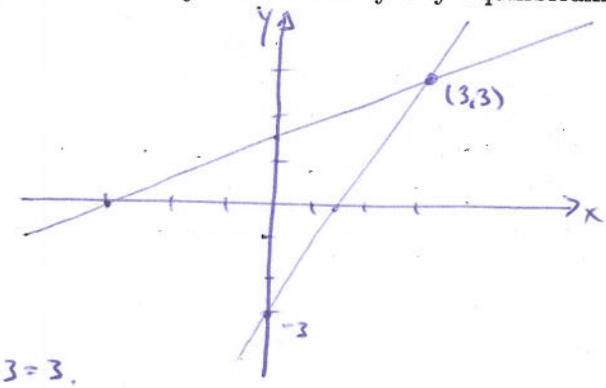
$y = \frac{x}{2} + \frac{3}{2}$

y-nullcline: $-2x + y + 3 = 0$

$y = 2x - 3$.

equilibrium: $\frac{x}{2} + \frac{3}{2} = 2x - 3$

$\frac{9}{2} = \frac{3}{2}x \Rightarrow x = 3,\ y = 2x - 3 = 3.$



(b) [4 points] Find a second-order differential equation that is equivalent to this first-order system of differential equations.

Option 1: $2y = x + 3 - x'$

$y = \frac{1}{2}(x + 3 - x')$

$y' = \frac{1}{2}(x' - x'')$

$\frac{1}{2}(x' - x'') = -2x + \frac{1}{2}(x + 3 - x') + 3$

$-\frac{1}{2}x'' + 0x' + \frac{3}{2}x = \frac{9}{2}$

$x'' - 2x' - 3x = -9.$

Option 2: $2x = y + 3 - y'$

$x = \frac{1}{2}(y + 3 - y')$

$x' = \frac{1}{2}(y' - y'')$

$\frac{1}{2}(y' - y'') = \frac{1}{2}(y + 3 - y') - 2y + 3$

$-\frac{1}{2}y'' + y' + \frac{3}{2}y = \frac{9}{2}$

$y'' - 2y' - 3y = -9$

(c) [8 points] Find the solution to the first-order system of differential equations in (c).

Homogeneous: $r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3,\ r = -1.$

$x_c = c_1 e^{3t} + c_2 e^{-t}$

Particular: $X = A$

$X' = 0,\ X'' = 0.$

$X'' - 2X' - 3X = 0 - 0 - 3A = -9$

$\Rightarrow A = 3.$

$x_p = 3.$

$x = c_1 e^{3t} + c_2 e^{-t} + 3$

Then $y = \frac{1}{2}(x + 3 - x')$

$= \frac{1}{2}\left(c_1 e^{3t} + c_2 e^{-t} + 3 + 3 - (3c_1 e^{3t} - c_2 e^{-t})\right)$

$= -c_1 e^{3t} + c_2 e^{-t} + 3$

Homogeneous: $r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3,\ r = -1$

$y_c = c_1 e^{3t} + c_2 e^{-t}$

Particular: $Y = A.\ (Y' = Y'' = 0)$

$Y'' - 2Y' - 3Y = 0 - 0 - 3A = -9 \Rightarrow A = 3.$

$y_p = 3.$

so $y = c_1 e^{3t} + c_2 e^{-t} + 3$

Then $x = \frac{1}{2}(y + 3 - y')$

$= \frac{1}{2}\left(c_1 e^{3t} + c_2 e^{-t} + 3 + 3 - (3c_1 e^{3t} - c_2 e^{-t})\right)$

$= -c_1 e^{3t} + c_2 e^{-t} + 3.$

6. (25 points) Let $A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) [20 points] Find the eigenvalues and eigenvectors for the matrix A.

$$\det(A-\lambda I) = \det\begin{pmatrix} -1-\lambda & 1 & 0 \\ -2 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} -1-\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix}$$
$$= (1-\lambda)\big((-1-\lambda)(2-\lambda) - (-2)(1)\big)$$
$$= (1-\lambda)(-2-2\lambda+\lambda+\lambda^2+2) = (1-\lambda)(-\lambda+\lambda^2) = \lambda(1-\lambda)(\lambda-1)$$
$$= -\lambda(1-\lambda)^2.$$

The eigenvalues are $\lambda=0$ and $\lambda=1$.

$\lambda=0$: $(A-0I)x=0$: $\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[③+①\to③]{②-2①\to②} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$\begin{aligned} -x_1+x_2 &= 0 \Rightarrow x_1 = x_2 \\ x_2+x_3 &= 0 \Rightarrow x_3 = -x_2 \end{aligned} \quad \text{so} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ -x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

An eigenvector of A for $\lambda=0$ is $x = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$\lambda=1$: $(A-I)x=0$: $\begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[③\cdot2+①\to③]{②-①\to②} \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$\begin{aligned} -2x_1+x_2 &= 0 \rightarrow x_1 = \tfrac{x_2}{2} = 0. \\ x_2 &= 0 \end{aligned} \quad \text{So} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

An eigenvector of A for $\lambda=1$ is $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Note: A is deficient.

(b) [Extra Credit: 5 points] How many linear trajectories are there for $x' = Ax$?

There is one linear trajectory for each linearly independent. eigenvector of A. This matrix has 2 eigenvectors, so there will be 2 linear trajectories.

for a real eigenvalue.