MATH 520 (Section 001)
Prof. Meade

University of South Carolina
Spring 2008

Exam 1
February 8, 2008

Name: Key
SS # (last 4 digits): ______

Instructions:

1. There are a total of 6 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	30	
2	15	
3	15	
4	20	
5	10	
6	10	
Total	100	

Good Luck!

1. (30 points) For each of the following differential equations (DEs):

   (i) determine if the DE is separable

   (ii) if the DE is separable, find the general solution to the DE.

   (iii) if an initial condition is given, find the solution to the IVP.

   (a) $\dfrac{dy}{dt} + y = e^t$

$$\frac{dy}{dt} = e^t - y$$

not separable

   (b) $\dfrac{dy}{dt} + ty = t$, $y(0) = 3$

$$\frac{dy}{dt} = t - ty = (1-y)t$$

$$\int \frac{dy}{1-y} = \int t\,dt \quad \begin{pmatrix} u = 1-y \\ du = -dy \end{pmatrix}$$

$$-\ln|1-y| = \tfrac{1}{2}t^2 + C$$

$$\ln|1-y| = -\tfrac{1}{2}t^2 + C$$

$$e^{\ln|1-y|} = e^{-\frac{1}{2}t^2 + C}$$

$$|1-y| = e^C e^{-\frac{1}{2}t^2}$$

$$1 - y = C e^{-\frac{1}{2}t^2}$$

$y(0) = 3$: $1 - 3 = Ce^{-\frac{1}{2}\cdot 0^2}$

$$-2 = C$$

so $1 - y = -2e^{-\frac{1}{2}t^2}$

and $y = 1 + 2e^{-\frac{1}{2}t^2}.$

   (c) $y^2 \dfrac{dy}{dx} = \dfrac{\ln(x)}{x + xy^3} = \dfrac{\ln(x)}{(1+y^3)x}$

$$\int (1+y^3)y^2\,dy = \int \frac{\ln(x)}{x}\,dx \qquad \begin{matrix} u = \ln(x) \\ du = \frac{1}{x}dx \end{matrix}$$

$$\int y^2 + y^5\,dy = \int u\,du$$

$$\tfrac{1}{3}y^3 + \tfrac{1}{6}y^6 = \tfrac{1}{2}u^2 + C. \longleftarrow$$ general sol'n in implicit form.

Note: To find the explicit solution, let $v = y^3$. Then $\frac{1}{3}v + \frac{1}{6}v^2 = \frac{1}{2}u^2 + C$ & use the quadratic formula to solve for v, then take cube roots.

2. (15 points) Consider the differential equation

$$\frac{dy}{dt} = \frac{\sqrt{25 - y^2}}{1+t}.$$

State the region in the $t - y$ plane where the Theorem about Existence of Unique Solutions to a First-Order DE (Theorem 2.4.1) guarantees a unique solution through any given initial point in the region.

$$f(t,y) = \frac{\sqrt{25-y^2}}{1+t}$$

continuous for all (t,y) with $25-y^2 \geq 0$, $1+t \neq 0$.

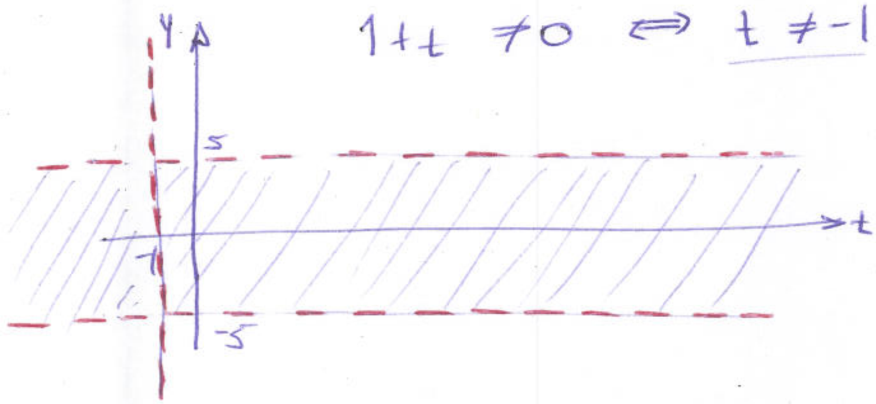
$$\frac{\partial f}{\partial y} = \frac{\frac{1}{2}(25-y^2)^{-1/2}(-2y)}{1+t} = \frac{-y}{(25-y^2)^{1/2}(1+t)}$$

continuous for all (t,y) with $25-y^2 > 0$, $1+t \neq 0$.

So, the Theorem guarantees the existence of a unique solution to this problem for any initial condition (t,y) with

$$25-y^2 > 0 \iff y^2 < 25 \iff |y| < 5$$

$$1+t \neq 0 \iff t \neq -1.$$



3. (15 points) Use the Theorem about Existence of Unique Solutions to Linear DEs (Theorem 2.4.2) to determine the guaranteed interval of existence for solutions to

$$(x-4)(x+2)\frac{d^2y}{dx^2} + \frac{x}{e^x}\frac{dy}{dx} + y = \sin(x)$$

with initial conditions given at

$$y'' + \frac{x}{(x-4)(x+2)e^x}\,y' + \frac{1}{(x-4)(x+2)}\,y = \frac{\sin(x)}{(x-4)(x+2)}$$

coefficients are not continuous when $x=4$ or $x=-2$.

(a) $x = 3$

the theorem guarantees a solution for all $-2 < x < 4$.

(b) $x = -2$

The theorem does not apply ~~at~~ when the initial conditions are given at $x = -2$.

(c) $x = -3$

the theorem guarantees a unique solution for all $-\infty < x < -2$.

4. (20 points) Eight differential equations and four slope fields are given below.

(a) [4 points] Find the zero isoclines for differential equation (viii).

$y' = y^2 - t$ $\quad$ $y^2 = t$ $\quad$ or $\quad$ $y = \pm\sqrt{t}$

$0 = y^2 - t$ $\quad$ (a parabola opening to the right)

(b) [4 points] Draw the solution curve through $(0, 2)$ on slope field (D). (see graph - in red)

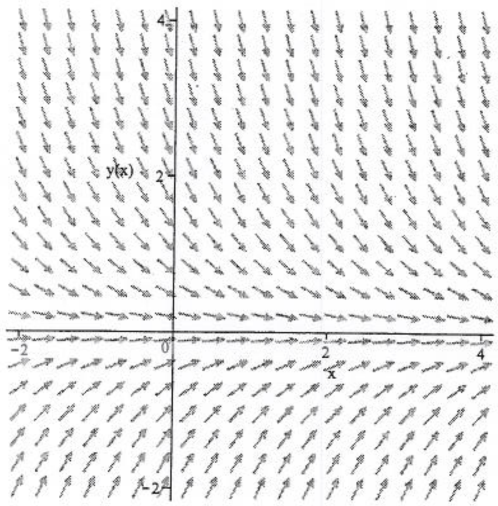
(c) [12 points] Determine the differential equation that corresponds to each slope field.

(i)	$y' = 1 - y^2$	(ii)	$y' = t - y$	(iii)	$y' = 1 - y$	(iv)	$y' = 1 - t$
(v)	$y' = -y$	(vi)	$y' = -y^2$	(vii)	$y' = y^2$	(viii)	$y' = y^2 - t$

autonomous: (i) (iii) (v) (vi) (vii)

equilibrium: $y = 0$ (v) (vi) (vii)

$\frac{dy}{dt} < 0$ for $y > 0$, $\frac{dy}{dt} > 0$ for $y < 0$ } (v)



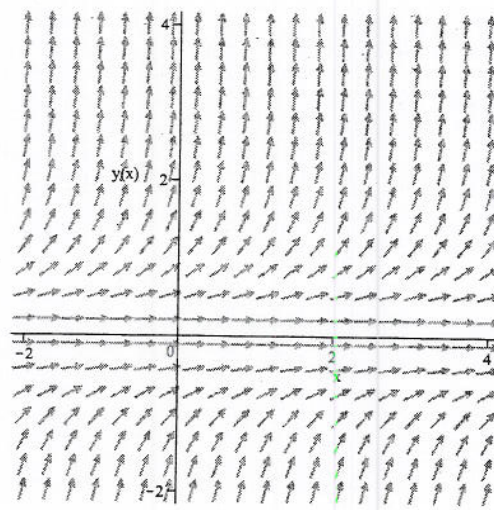
(A) (v)

autonomous

equil. sol: $y = 0$

$\frac{dy}{dt} > 0$ for $y \neq 0$.

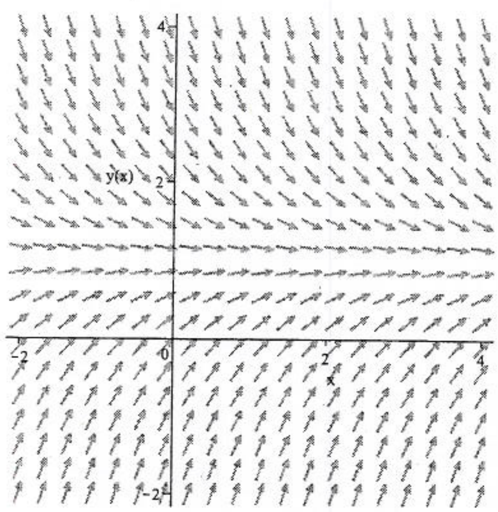
(vii)



(B) (vii)

autonomous:

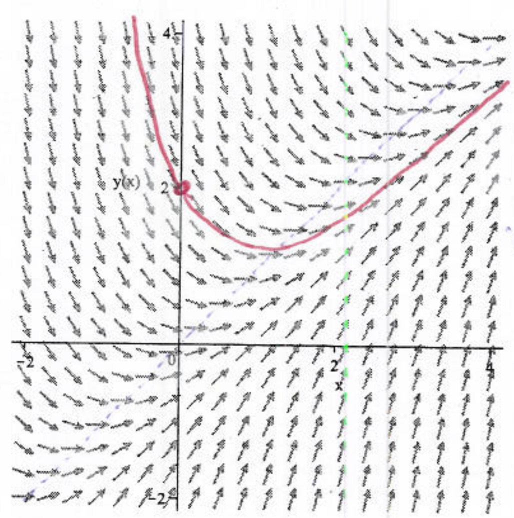
equil: $y = 1$: (iii)



(C) (iii)

non-autonomous (ii) (iv) (viii)

isocline: $y = t$ (ii)



(D) (ii)

5. (10 points) Consider the initial value problem

$$\frac{dy}{dt} = t - 0.2y^2, \qquad y(0) = 1.$$

$n=2$

$\Delta t = \frac{1-0}{n} = \frac{1}{2} = 0.5$

Use Euler's method with 2 steps to approximate $y(1)$.

NOTE: *The arithmetic does not get very messy. Use decimals, not fractions.*

i	t_i	y_i	$m_i = f(t_i, y_i)$	$y_{i+1} = y_i + m_i \Delta t$
0	0	1	$0 - 0.2(1)^2$ $= -0.2$	$1 + (-0.2)(0.5)$ $= 1 - 0.1$ $= 0.9$
1	0.5	0.9	$0.5 - 0.2(0.9)^2$ $= 0.5 - 0.2(0.81)$ $= 0.5 - 0.162$ $= 0.338$	$0.9 + 0.338(0.5)$ $= 0.9 + 0.169$ $= 1.069$
2	1.0	1.069		

$y(1) \approx 1.069$

6. (10 points) Let $p = p(t)$ be the size of a population at time t. Write the differential equation corresponding to the following description:

The relative rate of change of the population is proportional to the square of the difference between the fixed capacity of the environment, N, and the current size of the population.

$$\underbrace{\frac{1}{P}\frac{dp}{dt}}_{\text{relative rate of change}} \underbrace{= k \cdot}_{\text{is proportional to}} \underbrace{(N-p)^2}_{\text{square of difference}}$$