HW Solutions - #9

§4.1 #6. $(x^2-4)\,y^{(6)} + x^2 y''' + 9y = 0$

in standard form: $y^{(6)} + \frac{x^2}{(x-2)(x+2)} y''' + \frac{9}{(x-2)(x+2)} y = 0$.

The coefficients are continuous, except at $x = 2$ and $x = -2$.

Solutions are sure to exist on $(-\infty, -2)$, $(-2, 2)$, or $(2, \infty)$.

#15. $xy''' - y'' = 0$

$y_1 = 1: \ y_1' = y_1'' = y''' = 0 \qquad x \cdot 0 - 0 = 0$ ✓

$y_2 = x: \ y_2' = 1, \ y_2'' = y_2''' = 0 \qquad x \cdot 0 - 0 = 0$ ✓

$y_3 = x^3: \ y_3' = 3x^2, \ y_3'' = 6x, \ y_3''' = 6 \qquad x \cdot 6 - 6x = 0$ ✓

$$W[1, x, x^3] = \det \begin{bmatrix} 1 & x & 3x^2 \\ 0 & 1 & 6x \\ 0 & 0 & 6 \end{bmatrix} = 6 \neq 0.$$

§4.2 #14. $y^{(4)} - 4y''' + 4y'' = 0$

$y = e^{rt}: \ r^4 - 4r^3 + 4r^2 = r^2(r^2 - 4r + 4) = r^2(r-2)^2 = 0$

$r = 0, 0, 2, 2.$

$$\boxed{y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}.}$$

#17. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$

$y = e^{rt}: \ r^6 - 3r^4 + 3r^2 - 1 \underset{(s = r^2)}{=} s^3 - 3s^2 + 3s - 1 = (s-1)^3 = (r^2-1)^3 = (r-1)^3(r+1)^3 = 0$

$r = 1, 1, 1, -1, -1, -1.$

$$\boxed{y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 e^{-t} + c_5 t e^{-t} + c_6 t^2 e^{-t}.}$$

#20. $y^{(4)} - 8y' = 0$

$y = e^{rt}: \ r^4 - 8r = r(r^3 - 8) = r(r-2)(r^2 + 2r + 4) = 0$

$r = 0, \ r = 2, \ r = \frac{1}{2}\left(-2 \pm \sqrt{4 - 16}\right) = \frac{1}{2}(-2 \pm 2\sqrt{3}\,i) = -1 \pm \sqrt{3}\,i.$

$$\boxed{y = c_1 + c_2 e^{2t} + c_3 e^{-t}\cos(\sqrt{3}\,t) + c_4 e^{-t}\sin(\sqrt{3}\,t).}$$

#31. $y^{(4)} - 4y''' + 4y'' = 0, \quad y(1) = -1, \ y'(1) = 2, \ y''(1) = 0, \ y'''(1) = 0.$

$y = e^{rt}: \ r^4 - 4r^3 + 4r^2 = r^2(r^2 - 4r + 4) = r^2(r-2)^2 = 0$

$r = 0, 0, 2, 2$

$y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$

$y' = c_2 + 2c_3 e^{2t} + c_4 e^{2t} + 2c_4 t e^{2t} = c_2 + (2c_3 + c_4)e^{2t} + 2c_4 t e^{2t}$

$y'' = 4c_3 e^{2t} + 2c_4 e^{2t} + 2c_4 e^{2t} + 4c_4 t e^{2t} = (4c_3 + 4c_4)e^{2t} + 4c_4 t e^{2t}$

$y''' = 8c_3 e^{2t} + 4c_4 e^{2t} + 4c_4 e^{2t} + 4c_4 e^{2t} + 8c_4 t e^{2t} = (8c_3 + 12c_4)e^{2t} + 8c_4 t e^{2t}$

$y(1) = c_1 + c_2 + c_3 e^2 + c_4 e^2 = -1$

$y'(1) = c_2 + 3c_3 e^2 + 2c_4 e^2 = 2$

$y''(1) = 4c_3 e^2 + 8c_4 e^2 = 0 \Rightarrow c_3 + 2c_4 = 0$

$y'''(1) = 8c_3 e^2 + 20c_4 e^2 = 0 \rightarrow 2c_3 + 5c_4 = 0$

$\overset{①}{\Rightarrow} c_3 = c_4 = 0$

$\overset{②}{\Rightarrow} c_2 = 2$

$\overset{③}{\Rightarrow} c_1 = -1 - 2 = -3$

$$\therefore \boxed{y = -3 + 2t}$$

§4.3 #7. $y^{(6)}+y'''=t$

Homog: $y=e^{rt}$: $r^6+r^3=0$

$r^3(r^3+1)=r^3(r+1)(r^2-r+1)=0$

$r=0,0,0,-1,\frac{1}{2}\left(1\pm\sqrt{1-4}\right)=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y_h = c_1+c_2t+c_3t^2+c_4e^{-t}+c_5e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+c_6e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t.$

Particular: $y_p = t^3(At+B) = At^4+Bt^3$

$y_p' = 4At^3+3Bt^2$ $\quad y_p^{(4)} = 24A$

$y_p'' = 12At^2+6Bt$ $\quad y_p^{(5)} = 0$

$y_p''' = 24At+6B$ $\quad y_p^{(6)} = 0.$

$0+(24At+6B)=t \Longrightarrow$ t: $24A=1 \Rightarrow A=1/24$

1: $6B=0 \Rightarrow B=0$

$\boxed{y_p=\frac{1}{24}t^4}$

$y = c_1+c_2t+c_3t^2+c_4e^{-t}+c_5e^{\frac{t}{2}}\cos\sqrt{3}/2\,t+c_6e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t+\frac{1}{24}t^4.$

#10, $y^{(4)}+2y''+y=3t+4$ $\quad y(0)=y'(0)=0,\ y''(0)=y'''(0)=1.$

Homog: $y=e^{rt}$: $r^4+2r^2+1=(r^2+1)^2=0.$

$r=\pm i,\pm i.$

$y_h = c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t$

Particular: $y_p=At+B$ $\quad y_p''=y_p'''=y_p^{(4)}=0.$

$y_p'=A$

$0+2(0)+At+B=3t+4 \Rightarrow$ t: $A=3$

1: $B=4$

$y_p=3t+4.$

$y=c_1\cos t+c_2\sin(t)+c_3t\cos(t)+c_4t\sin(t)+3t+4.$

$y'=-c_1\sin t+c_2\cos t+c_3\cos t-c_3t\sin t+c_4\sin t+c_4t\cos(t)+3$

$=(-c_1+c_4)\sin t+(c_2+c_3)\cos t-c_3t\sin t+c_4t\cos(t)+3$

$y''=(-c_1+c_4)\cos t-(c_2+c_3)\sin t-c_3\sin t-c_3t\cos t+c_4\cos t-c_4t\sin t$

$=(-c_1+2c_4)\cos t+(-c_2-2c_3)\sin t-c_3t\cos t-c_4t\sin t$

$y'''=(c_1-2c_4)\sin t+(-c_2-2c_3)\cos t-c_3\cos t+c_3t\sin t-c_4\sin t-c_4t\cos t$

$=(c_1-3c_4)\sin t+(-c_2-3c_3)\cos t+c_3t\sin t-c_4t\cos t$

$c_1+\cancel{c_3}=0 \Rightarrow c_1=0$ $\quad \Rightarrow c_2=1/2$

$c_2+c_3\cancel{+c_4}=0 \Rightarrow c_2+c_3=0$

$-c_1+2c_4=1 \Rightarrow 2c_4=1 \Rightarrow c_4=1/2$

$-c_2-3c_3=1 \Rightarrow -c_2-3c_3=1.$

$-2c_3=1 \Rightarrow c_3=-1/2$

$\therefore \boxed{y=\frac{1}{2}\sin t-\frac{1}{2}t\cos t+\frac{1}{2}t\sin t+3t+4}$

#16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$

Homog: $y = e^{rt}$: $r^4 + 4r^2 = r^2(r^2+4) = 0 \Rightarrow r = 0, 0, 2i, -2i.$

$y_h = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t$

Particular: $g_1 = \sin 2t$ $\quad y_{p_1} = t(A \cos 2t + B \sin 2t)$
$= At\cos 2t + Bt \sin 2t.$

$g_2 = te^t \quad y_{p_2} = (Ct + D)e^t$

$g_3 = 4 \quad y_{p_3} = t^2(E) = Et^2.$

so $\boxed{y_p = At\cos 2t + Bt \sin 2t + (Ct+D)e^t + Et^2.}$

Note: The interval should be $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

§4.4 #1. $y''' + y' = \tan t$

Homog: $y = e^{rt}$: $r^3 + r = r(r^2+1) = 0 \Rightarrow r = 0, r = i, r = -i$

$y_h = c_1 + c_2 \cos t + c_3 \sin t.$

Var-of Param: $y_p = v_1 \cancel{\cos t} + v_2 \cos t + v_3 \sin t.$

where v_1, v_2, v_3 satisfy:

$$\begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tan t. \end{bmatrix}.$$

so $v_1' + v_2' \cos t + v_3' \sin t = 0$

$-v_2' \sin t + v_3' \cos t = 0$

$-v_2' \cos t - v_3' \sin t = \tan t \quad \Longrightarrow v_1 = \ln|\sec t|$

① + ③ $\Rightarrow v_1' = \tan t$

$\sin t$ ② $+ \cos t$ ③ $\Rightarrow -v_2'(\sin t)^2 + v_3' \sin t \cos t = 0$

$+ \quad -v_2'(\cos t)^2 - v_3' \sin t \cos t = \sin t$

$-((\sin t)^2 + (\cos t)^2)\, v_2' = \sin t.$

$-v_2' = \sin t \Longrightarrow v_2 = \cos t$

$v_2' = -\sin t$

$v_2' \cancel{\sin t} \cos t = 0.$

② $\Rightarrow -(-\sin t)\sin t + v_3' \cos t = 0$

$\sin^2 t + \cos t\, v_3' = 0$

$v_3' = \frac{-\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t}$

$= \cos t - \sec t$

$\Longrightarrow v_3 = \sin t - \ln|\sec t + \tan t|$

$\therefore y_p = v_1 + v_2 \cos t + v_3 \sin t$

$= \ln|\sec t| + (\cos t)\cos t + (\sin t - \ln|\sec t + \tan t|)\sin t$

$= \ln|\sec t| + \underbrace{(\cos^2 t + \sin^2 t)}_{1 \leftarrow \text{in } y_h} - \sin t \ln|\sec t + \tan t|$

$= \ln|\sec t| + $

and the general solution is: $\boxed{y = c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec t| - \ln|\sec t + \tan t| \sin t}$

#13. $x^3y''' + x^2y'' - 2xy' + 2y = 2x^4 \quad (x>0).$

$y_1 = x, \; y_2 = x^2, \; y_3 = \frac{1}{x}.$

$y_p = v_1 x + v_2 \cdot x^2 + v_3 \cdot x^{-1}$ where

$$\begin{bmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}$$

remember to write (DE) in ~~___~~ standard form.

so

① $xv_1' + x^2v_2' + x^{-1}v_3' = 0$

② $v_1' + 2xv_2' - x^{-2}v_3' = 0$

③ $2v_2' + 2x^{-3}v_3' = 2x$

④ = ① − x②: $-x^2v_2' + 2x^{-1}v_3' = 0$

$x^2$③ + 2④: $(2x^{-1} + 4x^{-1})v_3' = 2x^3$

$$6x^{-1}v_3' = 2x^3$$

$$v_3' = \frac{1}{3}x^4 \implies v_3 = \frac{1}{15}x^5.$$

$$2v_2' + 2x^{-3}\left(\frac{1}{3}x^4\right) = 2x$$

$$2v_2' + \frac{2}{3}x = 2x$$

$$2v_2' = \frac{4}{3}x \implies v_2 = \frac{1}{3}x^2$$

$$v_2' = \frac{2}{3}x$$

$$\begin{aligned} v_1' &= -2xv_2' + x^{-2}v_3' \\ &= -2x\left(\tfrac{2}{3}x\right) + x^{-2}\left(\tfrac{1}{3}x^4\right) \\ &= -\tfrac{4}{3}x^2 + \tfrac{1}{3}x^2 \\ &= -x^2 \longrightarrow v_1 = -\tfrac{1}{3}x^3. \end{aligned}$$

$$\begin{aligned} \text{so } y_p &= v_1 \cdot x + v_2 \cdot x^2 + v_3 \cdot x^{-1} = \left(-\tfrac{1}{3}x^3\right)x + \left(\tfrac{1}{3}x^2\right)x^2 + \left(\tfrac{1}{15}x^5\right)\cdot x^{-1} \\ &= -\tfrac{1}{3}x^4 + \tfrac{1}{3}x^4 + \tfrac{1}{15}x^4 \\ &= \tfrac{1}{15}x^4 \end{aligned}$$

$$\therefore \boxed{y = c_1 x + c_2 x^2 + c_3 x^{-1} + \frac{1}{15}x^4}$$