HW Solutions – #8

§3.5 #1. $y'' - 2y' - 3y = 3e^{2t}$

Homog: $r^2 - 2r - 3 = (r-3)(r+1) = 0$

$y_1 = e^{3t} \quad y_2 = e^{-t}$

$y_h = c_1 e^{3t} + c_2 e^{-t}$

Guess: $y = Ae^{2t}$

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

$y'' - 2y' - 3y = 4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t})$

$= -3Ae^{2t} = 3e^{2t}$

$-3A = 3$

$y_p = -e^{2t} \Leftarrow \quad A = -1$

Gen'l solution: $y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$.

#15. $y'' - 2y' + y = te^t + 4$

Homog: $r^2 - 2r + 1 = (r-1)^2 = 0$

$y_1 = e^t \quad y_2 = te^t$

$y_h = c_1 e^t + c_2 te^t$

4: Guess: $y = A,\ y' = 0,\ y'' = 0$

$y'' - 2y' + y = 0 - 2(0) + A = \underline{A = 4}$

$y_{p_1} = 4$

te^t: Guess: $y = t^2(At + B)e^t = (At^3 + Bt^2)e^t = At^3e^t + Bt^2e^t$

$y' = At^3e^t + 3At^2e^t + Bt^2e^t + 2Bte^t$

$y'' = At^3e^t + 3At^2e^t + 3At^2e^t + 6Ate^t + Bt^2e^t + 2Bte^t + 2Bte^t + 2Be^t$

$y'' - 2y' + y = At^3e^t + 6At^2e^t + Bt^2e^t + 6Ate^t + 4Bte^t + 2Be^t$

$- 2(At^3e^t + 3At^2e^t + Bt^2e^t + 2Bte^t)$

$+ At^3e^t + Bt^2e^t$

$= 0At^3e^t + (0A + 0B)t^2e^t + (6A + 0B)te^t + 2Be^t$

$= 6Ate^t + 2Be^t = te^t \implies \begin{array}{l} 6A = 1 \\ 2B = 0 \end{array} \quad \begin{array}{l} A = 1/6 \\ B = 0 \end{array} \Rightarrow y_{p_2} = \frac{1}{6}t^3e^t$.

Gen'l solution: $y = c_1 e^t + c_2 te^t + 4 + \frac{1}{6}t^3e^t$.

$y' = c_1 e^t + c_2 te^t + c_2 e^t + \frac{1}{6}t^3e^t + \frac{1}{2}t^2e^t$

$y(0) = c_1 + 4 = 1 \Rightarrow c_1 = -3$

$y'(0) = c_1 + c_2 = 1 \Rightarrow c_2 = 1 -$

$= 1 +$

$= 4$

$\therefore y = -3e^t + 4te^t + 4 + \frac{1}{6}t^3e^t$.

#16. $y'' - 2y' - 3y = 3te^{2t}$

Homog: $r^2 - 2r - 3 = (r-3)(r+1) = 0$

$y_1 = e^{3t} \qquad y_2 = e^{-t}$

$y_h = c_1 e^{3t} + c_2 e^{-t}$

Guess: $y = (At+B)e^{2t} = Ate^{2t} + Be^{2t}$

$y' = 2Ate^{2t} + Ae^{2t} + 2Be^{2t}$

$y'' = 4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t} + 4Be^{2t} = 4Ate^{2t} + 4Ae^{2t} + 4Be^{2t}$

$$\begin{aligned} y'' - 2y' - 3y &= 4Ate^{2t} + 4Ae^{2t} + 4Be^{2t} \\ &\quad -2(2Ate^{2t} + Ae^{2t} + 2Be^{2t}) \\ &\quad -3(Ate^{2t} + Be^{2t}) \\ &= -3Ate^{2t} + 2Ae^{2t} - 3Be^{2t} = -3Ate^{2t} + (2A-3B)e^{2t} = 3te^{2t} \end{aligned}$$

$\Rightarrow te^{2t}: \ -3A = 3 \quad \Rightarrow A = -1$

$e^{2t}: \ 2A - 3B = 0 \quad \Rightarrow B = \frac{2A}{3} = \frac{2\cdot(-1)}{3} = -\frac{2}{3} \quad \Rightarrow y_p = -te^{2t} - \frac{2e^{2t}}{3}$

Gen'l Sol'n: $y = c_1 e^{3t} + c_2 e^{-t} - te^{2t} - \frac{2}{3}e^{2t}$

$y' = 3c_1 e^{3t} - c_2 e^{-t} - 2te^{2t} - e^{2t} - \frac{4}{3}e^{2t}$

$y(0) = c_1 + c_2 - \frac{2}{3} = 1 \quad \Rightarrow c_1 + c_2 = \frac{5}{3}$

$y'(0) = 3c_1 - c_2 - \frac{7}{3} = 0 \quad \Rightarrow 3c_1 - c_2 = 7/3$

$4c_1 = 4 \quad \Rightarrow c_1 = 1, \ c_2 = \frac{5}{3} - c_1 = \frac{5}{3} - 1 = \frac{2}{3}$

$\therefore \ \cancel{y = e^{3t} + \frac{2}{3}e^{-t} - te^{2t} - 2e^{2t}}$

$\cancel{y = e^{3t} + \frac{2}{3}e^{-t} - te^{2t} - 2e^{2t}}$

$y = e^{3t} + \frac{2}{3}e^{-t} - te^{2t} - \frac{2}{3}e^{2t}$

#17. $y'' + 4y = 3\sin 2t$

Homog: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$y_1 = \cos 2t \qquad y_2 = \sin 2t$

$y_h = c_1 \cos 2t + c_2 \sin 2t$

Guess: $y = t(A\cos 2t + B\sin 2t) = At\cos 2t + Bt\sin 2t$

$y' = -2At\sin 2t + A\cos 2t + 2Bt\cos 2t + B\sin 2t$

$y'' = -4At\cos 2t - 2A\sin 2t - 2A\sin 2t - 4Bt\sin 2t + 2B\cos 2t + 2B\cos 2t$

$= (-4At + 4B)\cos 2t + (-4A - 4Bt)\sin 2t$

$y'' + 4y = (-4At + 4B + 4At)\cos 2t + (-4A - 4Bt + 4Bt)\sin 2t$

$= 4B\cos 2t - 4A\sin 2t. = 3\sin 2t \Rightarrow \begin{cases} 4B = 0 \Rightarrow B = 0 \\ -4A = 3. \quad A = -3/4 \end{cases}$

$y_p = -\frac{3}{4}t\cos 2t$

Gen'l Sol'n: $y = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{4}t\cos 2t.$

$y' = -2c_1 \sin 2t + 2c_2 \cos 2t - \frac{3}{2}t\sin 2t - \frac{3}{4}\cos 2t$

$y(0) = c_1 = 2 \qquad \Rightarrow c_1 = 2$

$y'(0) = 2c_2 - \frac{3}{4} = -1 \qquad c_2 = \frac{1}{2}\left(-1 + \frac{3}{4}\right) = -\frac{1}{8}$

$\therefore y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t.$

#22a. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$

Homog: $r^2 + 2r + 2 = 0 \qquad r = \frac{1}{2}\left(-2 \pm \sqrt{2^2 - 4\cdot 2}\right) = \frac{1}{2}(-2 \pm 2i) = -1 \pm i$

$y_1 = e^{-t}\cos t \qquad y_2 = e^{-t}\sin t \qquad y_h = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$

$g_1(t) = 3e^{-t} : Y_{P_1} = Ae^{-t}$

$g_2(t) = 2e^{-t}\cos t + 4e^{-t}t^2\sin t : Y_{P_2} = t\left((Bt^2 + Ct + D)e^{-t}\cos t + (Et^2 + Ft + G)e^{-t}\sin t\right)$

$\therefore Y_p = Ae^{-t} + (Bt^3 + Ct^2 + Dt)e^{-t}\cos t + (Et^3 + Ft^2 + Gt)e^{-t}\sin t.$

#23a. $y'' - 4y' + 4 = 2t^2 + 4te^{2t} + t\sin 2t$

Homog: $r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow y_1 = e^{2t} \quad y_2 = te^{2t} \quad \therefore y_h = c_1 e^{2t} + c_2 te^{2t}$

$g_1(t) = 2t^2 : Y_{P_1} = At^2 + Bt + C$

$g_2(t) = 4te^{2t} : Y_{P_2} = t^2(Dt + E)e^{2t}$

$g_3(t) = t\sin t : Y_{P_3} = (Ft + G)\sin 2t + (Ht + I)\cos 2t$

$\therefore Y_p = At^2 + Bt + C + (Dt^3 + Et^4)e^{2t} + (Ft + G)\sin 2t + (Ht + I)\cos 2t.$

§3.6 #3. $y''+2y'+y=3e^{-t}$

Homog: $r^2+2r+1=(r+1)^2=0 \Rightarrow y_1=e^{-t}$, $y_2=te^{-t}$

Particular: $y_p=v_1e^{-t}+v_2te^{-t}$ where $\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t}+e^{-t} \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}=\begin{bmatrix} 0 \\ 3e^{-t} \end{bmatrix}$

$$e^{-t}v_1'+te^{-t}v_2'=0$$
$$-e^{-t}v_1'+(1-t)e^{-t}v_2'=3e^{-t}$$
$$e^{-t}v_2'=3e^{-t} \Rightarrow v_2'=3 \Rightarrow v_2=3t.$$
$$v_1'+tv_2'=0 \Rightarrow v_1'=-tv_2'=-3t \Rightarrow v_1=-\tfrac{3}{2}t^2$$

so $y_p=v_1y_1+v_2y_2=-\frac{3}{2}t^2e^{-t}+3t\,te^{-t}=\frac{3}{2}t^2e^{-t}$.

$\therefore\ y=c_1e^{-t}+c_2te^{-t}+\frac{3}{2}t^2e^{-t}$.

#14. $t^2y''-t(t+2)y'+(t+2)y=2t^3$, $t>0$.

$y_1=t$: $y_1'=1$, $y_1''=0$ so $t^2\cdot 0-t(t+2)1+(t+2)t=0$ ✓

$y_2=te^t$: $y_2'=(t+1)e^t$, $y_2''=(t+2)e^t$ so $t^2(t+2)e^t-t(t+2)(t+1)e^t+(t+2)te^t$

$=(t^2-t(t+1)+t)(t+2)e^t=0\cdot(t+2)e^t=0$ ✓

so $y_h=c_1t+c_2te^t$.

Particular: Note: standard form is $y''-\frac{t+2}{t}y'+\frac{t+2}{t^2}y=2t$

$y_p=v_1t+v_2te^t$ where $\begin{bmatrix} t & te^t \\ 1 & (t+1)e^t \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}=\begin{bmatrix} 0 \\ 2t \end{bmatrix}$

$$-(\cancel{t}v_1'+\cancel{t}e^tv_2'=0)$$
$$v_1'+(t+1)e^tv_2'=2t$$
$$(-1+1)v_1'+(-1+t+1)e^tv_2'=2t \Rightarrow te^tv_2'=2t \Rightarrow v_2'=2e^{-t} \Rightarrow v_2=-2e^{-t}$$
$$v_1'+e^tv_2'=0 \Rightarrow v_1'=-e^tv_2'=-e^t(2e^{-t})=-2 \Rightarrow v_1=-2t$$

so $y_p=(-2t)t+(-2e^{-t})te^t=-2t^2-2t$

$\therefore\ y=c_1t+c_2te^t-2t^2-2t$

$=(c_1-2)t+c_2te^t-2t^2$

$=c_1t+c_2te^t\underline{-2t^2}$

Note that $-2t$ is a solution to the homogeneous equation, so it adds nothing to the particular solution—and can be ignored.