Solutions - HW2

§1.2 #1. (a) $\frac{dy}{dt} = -y+5$, $y(0) = y_0$.

Separable: $\frac{dy}{y-5} = -dt$

$\ln|y-5| = -t + C$

$|y-5| = Ce^{-t}$

$y = 5 + Ce^{-t}$

$y(0) = y_0$: $y_0 = 5 + C$

$C = y_0 - 5$

$\boxed{y(t) = 5 + (y_0 - 5)e^{-t}}$

$\lim_{t\to\infty} y = 5$ for all values of y_0

(b) $\frac{dy}{dt} = -2y+5$, $y(0) = y_0$

By the same process: $\boxed{y = \frac{5}{2} + (y_0 - \frac{5}{2})e^{-2t}}$

$\lim_{t\to\infty} y \to \frac{5}{2}$ for all values of y_0.

(c) $\frac{dy}{dt} = -2y+10$, $y(0) = y_0$

By the same process: $\boxed{y = 5 + (y_0 - 5)e^{-2t}}$

$\lim_{t\to\infty} y = 5$ for all values of y_0.

Note: The solutions in (b) & (c) converge to their equilibrium faster than the solution in (a).

#4. $\frac{dy}{dt} = -ay+b$.

(a) Equilibrium sol'n: $y(t) = y_e$ so $y' = 0$.

$0 = -ay_e + b \Rightarrow ay_e = b \Rightarrow \boxed{y_e = \frac{b}{a}.}$

(b) $Y(t) = y(t) - y_e$: $\frac{dY}{dt} = \frac{d}{dt}(y - y_e) = \frac{dy}{dt} = -ay+b$

so $y = Y + y_e$. $= -a(Y+y_e)+b = -aY + \underbrace{(-ay_e+b)}_{0} = -aY$.

$\therefore \boxed{\frac{dY}{dt} = -aY}$

#7. $\frac{dp}{dt} = 0.5p - 450$ Gen'l sol'n: $p = 900 + (p_0 - 900)e^{t/2}$.

(a) with $p_0 = 850$: $p(t) = 900 - 50e^{t/2}$

Extinct when $p(t) = 0$: $900 - 50e^{t/2} = 0$

$900 = 50e^{t/2}$

$18 = e^{t/2}$

$\ln 18 = t/2$

$\boxed{t = 2\ln 18.}$

(b) For a general p_0: $900 + (p_0 - 900)e^{t/2} = 0$

Note: $\frac{900}{900-p_0} > 0$ when $0 < p_0 < 900$

$900 = (900 - p_0)e^{t/2}$

$\frac{900}{900-p_0} = e^{t/2}$

$\ln\left(\frac{900}{900-p_0}\right) = \frac{t}{2} \Rightarrow \boxed{t = 2\ln\left(\frac{900}{900-p_0}\right)}$

(c) Extinct in 1 yr $\Rightarrow p(12) = 0$: $900 + (p_0 - 900)e^{6} = 0$

$900 = (900 - p_0)e^{6}$

$900e^{-6} = 900 - p_0$

$\boxed{p_0 = 900 - 900e^{-6}}$

§1.3 #1. 2^{nd} order, linear

2. 2^{nd} order, nonlinear (y^2)

3. 4^{th} order, linear

4. 1^{st} order, nonlinear (y^2)

5. 2^{nd} order, nonlinear $(\sin(t+y))$

6. 3^{rd} order, linear

#8. $y''+2y'-3y=0$: $y_1 = e^{-3t}$ $\quad y_1' = -3e^{-3t}$ $\quad y_1'' = 9e^{-3t}$

$$y_1''+2y_1'-3y = 9e^{-3t}+2\left(-3e^{-3t}\right)-3\left(e^{-3t}\right) =$$

$$=(9-6-3)e^{-3t} = 0$$

$$y_2 = e^t \quad y_2' = e^t \quad y_2'' = e^t$$

$$y_2''+2y_2'-3y_2 = e^t+2(e^t)-3(e^t) = (1+2-3)e^t = 0.$$

#11 $2t^2y''+3ty'-y=0$:

$$y_1 = t^{1/2} \quad y_1' = \frac{1}{2}t^{-1/2} \quad y_1'' = -\frac{1}{4}e^{-3/2}$$

$$2t^2y_1''+3ty_1'-y_1 = 2t^2\left(-\frac{1}{4}t^{-3/2}\right)+3t\left(\frac{1}{2}t^{-1/2}\right)-t^{1/2}$$

$$= \left(-\frac{1}{2}t^{1/2}\right)+\frac{3}{2}t^{1/2}-t^{1/2} = \left(-\frac{1}{2}+\frac{3}{2}-1\right)t^{1/2} = 0.$$

$$y_2 = t^{-1} \quad y_2' = -t^{-2} \quad y_2'' = 2t^{-3}$$

$$2t^2y_2''+3ty_2'-y_2 = 2t^2\left(2t^{-3}\right)+3t\left(-t^{-2}\right)-\left(t^{-1}\right) = 4t^{-1}-3t^{-1}-t^{-1} = 0.$$

#14. $y'-2ty=1$:

$$y = e^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}$$

$$y' = e^{t^2}\frac{d}{dt}\int_0^t e^{-s^2}ds + 2te^{t^2}\int_0^t e^{-s^2}ds + 2te^{t^2}$$

$$= \underbrace{e^{t^2}\left(e^{-t^2}\right)}_{1} + 2te^{t^2}\int_0^t e^{-s^2}ds + 2te^{t^2}$$

so

$$y'-2ty = 1+2te^{t^2}\int_0^t e^{-s^2}ds+2te^{t^2}-2t\left(e^{t^2}\int_0^t e^{-s^2}ds+e^{t^2}\right)$$

$$= 1$$

§2.2 #7. $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$

Separable: $(y+e^{y})\,dy = (x-e^{-x})\,dx$

$$\boxed{\tfrac{1}{2}y^2+e^{y} = \tfrac{1}{2}x^2+e^{-x} + C.}$$

#2. $y' = \frac{x^2}{y(1+x^3)}$

Separable: $y\,dy = \frac{x^2}{1+x^3}\,dx$

$$\boxed{\tfrac{1}{2}y^2 = \tfrac{1}{3}\ln|1+x^3| + C}$$

#10(a) $y' = \frac{1-2x}{y}$, $y(1) = -2$:

Separable: $y\,dy = (1-2x)\,dx$

$\tfrac{1}{2}y^2 = x - x^2 + c.$

$y(1) = -2$: $\tfrac{1}{2}(-2)^2 = 1 - 1 + c \Rightarrow c = 2$: $\tfrac{1}{2}y^2 = x - x^2 + 2$

$y^2 = 4 + 2x - 2x^2$

$y = -\sqrt{4+2x-2x^2}$

(c) Soln exists when $4+2x-2x^2 \geq 0$ ← can't have $y=0$

$2(2+x-x^2) > 0$

$2(2-x)(1+x) > 0 \iff \boxed{-1 < x < 2}$

#13. $y' = \frac{2x}{y+x^2y}$, $y(0) = -2$

(a) Separable: $\frac{dy}{dx} = \frac{2x}{y(1+x^2)}$ $\quad y\,dy = \frac{2x}{1+x^2}\,dx$

$\tfrac{1}{2}y^2 = \ln(1+x^2) + c.$

$y(0) = -2$: $\tfrac{1}{2}(-2)^2 = \ln(1) + c \Rightarrow c = 2$

$\tfrac{1}{2}y^2 = \ln(1+x^2) + 2$

$y^2 = 2\ln(1+x^2) + 4$

$y = -\sqrt{2\ln(1+x^2)+4}$

(c) This solution exists for all values of x

because $1+x^2 \geq 1$ and $\ln(1+x^2) \geq 0$ so that $2\ln(1+x^2)+4 \geq 4 > 0$.

#24. $y' = \dfrac{2-e^x}{3+2y}$, $y(0)=0$

Separable: $(3+2y)\,dy = (2-e^x)\,dx$

$$3y+y^2 = 2x - e^x + C$$

$y(0)=0$: $3\cdot 0+0^2 = 2\cdot 0 - e^0 + C$

$0 = -1 + C \Rightarrow C = +1 \quad \therefore \boxed{3y+y^2 = 2x - e^x + 1}$

A maximum must occur at a critical point, i.e. where $y'=0$

$y'=0 \Leftrightarrow \dfrac{2-e^x}{3+2y} = 0 \Leftrightarrow 2-e^x = 0 \Leftrightarrow 2 = e^x \Leftrightarrow \boxed{x = \ln 2.}$

#31. (a) $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2} = 1 + \dfrac{y}{x} + \dfrac{y^2}{x^2} = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2 \Rightarrow$ homogeneous.

(b) Let $v = \dfrac{y}{x}$, so $y = xv$ and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

$x\dfrac{dv}{dx} + v = \dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2 = 1+v+v^2$ so $x\dfrac{dv}{dx} + v = 1+v+v^2$

separable. $\longrightarrow$ $x\dfrac{dv}{dx} = 1+v^2$

$$\frac{dv}{1+v^2} = \frac{dx}{x}$$

$\cancel{\arctan(1+v^2) = \ln|x| + C}$ $\quad$ $\arctan v = \ln|x| + C$

$\cancel{\arctan\left(1+\left(\frac{y}{x}\right)\right)}$ $\quad$ $\arctan\left(\dfrac{y}{x}\right) = \ln|x| + C$ $\leftarrow$ also ok

$$\frac{y}{x} = \tan(\ln|x| + c)$$

$$\boxed{y = x\tan(\ln|x| + c)}$$