HW13 - Solutions

§7.5 #4. $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$.

a) $\det(A - rI) = \det \begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} = (1-r)(-2-r) - 4 = r^2 + r - 6 = (r+3)(r-2) = 0$

$r = 2$: $(A - 2I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ so $-\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = \xi_1$

$\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

$r = -3$: $(A + 3I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$ so $4\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = -4\xi_1$

$\vec{\xi} = \begin{pmatrix} \xi_1 \\ -4\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ so $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$.

General solution: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$.

#17. $\vec{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

$\det(A - rI) = \det \begin{pmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{pmatrix} = (1-r)\Big((2-r)(3-r) - 2\Big) - 0 \cdot (\;) + (-1)\Big(2 - 2(2-r)\Big)$

$= (1-r)(r^2 - 5r + 4) - (2r - 2) = (1-r)(r-4)(r-1) + 2(1-r)$

$= (1-r)(r^2 - 5r + 4 + 2) = (1-r)(r^2 - 5r + 6) = (1-r)(r-2)(r-3)$

$r = 1$: $(A - 1I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

so $-\xi_1 = 0$ & $\xi_2 + 2\xi_3 = 0 \Rightarrow \xi_2 = -2\xi_3$ $\therefore$ $\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\xi_3 \\ \xi_3 \end{pmatrix} = \xi_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

$r = 2$: $(A - 2I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

so $-\xi_1 + \xi_2 = 0$ & $\xi_3 = 0 \Rightarrow \xi_1 = \xi_2$, $\xi_3 = 0$ $\therefore$ $\vec{\xi} = \begin{pmatrix} \xi_2 \\ \xi_2 \\ 0 \end{pmatrix} = \xi_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$r = 3$: $(A - 3I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

so $-\xi_1 + \xi_2 = 0$ & $-\xi_2 + 2\xi_3 = 0 \Rightarrow \xi_1 = \xi_2$ & $\xi_3 = \frac{1}{2}\xi_2$ $\therefore$ $\vec{\xi} = \begin{pmatrix} \xi_2 \\ \xi_2 \\ \frac{1}{2}\xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}$

General solution: $\vec{x}(t) = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix} e^{3t}$.

To satisfy the initial condition: $\vec{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{x}(0) = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & 1/2 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \therefore \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

$$\vec{x}(t) = 1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^t + 2e^{2t} \\ e^t \end{pmatrix}.$$

§7.6 #3. $\vec{x}\,' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$

a). $\det(A - rI) = \det\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} = (2-r)(-2-r) + 5 = r^2 + 1 = 0$

$r = +i$: $(A - iI)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2-i \\ 2-i & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2-i \\ 0 & 0 \end{pmatrix}$

$-5 - (2-i)(-2-i) = -5 - (-4-1) = 0$

so $\xi_1 + (-2-i)\xi_2 = 0 \Rightarrow \xi_1 = (2+i)\xi_2 \quad \therefore \vec{\xi} = \begin{pmatrix} (2+i)\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$

$$\vec{x} = \vec{\xi} e^{rt} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 2\cos t + 2i\sin t + i\cos t - \sin t \\ \cos t + i \sin t \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

so $\vec{x}^{(1)} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$ & $\vec{x}^{(2)} = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

General solution: $\vec{x} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$

#6. $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \vec{x}$

a) $\det(A - rI) = \det\begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix} = (1-r)(-1-r) + 10 = r^2 + 9 = 0.$

$r = 3i$: $(A - 3iI)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \xrightarrow[\textcircled{2} \to \textcircled{2}]{5\textcircled{1} + (1-3i)} \begin{pmatrix} 1-3i & 2 \\ 0 & 0 \end{pmatrix}$

so $(1-3i)\xi_1 + 2\xi_2 = 0 \Rightarrow \xi_2 = -\frac{1}{2}(1-3i)\xi_1 \therefore \vec{\xi} = \xi_1 \begin{pmatrix} 1 \\ -\frac{1}{2}(1-3i) \end{pmatrix}$

$\vec{x} = \vec{\xi} e^{rt} = \begin{pmatrix} 1 \\ -\frac{1}{2}(1-3i) \end{pmatrix} e^{i3t} = \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{3}{2}i \end{pmatrix} (\cos 3t + i \sin 3t)$

$= \begin{pmatrix} \cos 3t + i \sin 3t \\ -\frac{1}{2}\cos 3t - \frac{i}{2}\sin 3t + \frac{3i}{2}\cos 3t - \frac{3}{2}\sin 3t \end{pmatrix}$

$= \begin{pmatrix} \cos 3t \\ -\frac{1}{2}\cos 3t - \frac{3}{2}\sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ -\frac{1}{2}\sin 3t + \frac{3}{2}\cos 3t \end{pmatrix}$

The general solution is $\vec{x} = c_1 \begin{pmatrix} \cos 3t \\ -\frac{1}{2}\cos 3t - \frac{3}{2}\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t \\ -\frac{1}{2}\sin 3t + \frac{3}{2}\cos 3t \end{pmatrix}.$

§7.8 #2. $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x}$

c) $\det(A - rI) = \det\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix} = (4-r)(-4-r) + 16 = r^2 = 0 \Rightarrow r = 0, 0.$

$r = 0$: $(A - 0I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \to \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}$ so $4\xi_1 - 2\xi_2 = 0.$ $\Rightarrow \xi_2 = 2\xi_1.$

There is only one e-vector: $\vec{\xi} = \begin{pmatrix} \xi_1 \\ 2\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{0t} + \vec{\eta} e^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \vec{\eta}$ where $(A - 0I)\vec{\eta} = \vec{\xi}.$ $(\vec{\eta} \neq \vec{0})$

$\left(\begin{array}{cc|c} 4 & -2 & 1 \\ 8 & -4 & 2 \end{array}\right) \longrightarrow \begin{pmatrix} 4 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so $4\eta_1 - 2\eta_2 = 1.$ $\Rightarrow \eta_2 = 2\eta_1 - \frac{1}{2}$

so $\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 - \frac{1}{2} \end{pmatrix} = \eta_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$ & we choose $\vec{\eta} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

and so $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}.$

The general solution is $\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-1/2\end{pmatrix}$.

#9. $\vec{x}' = \begin{pmatrix}2 & \frac{3}{2}\\ -\frac{3}{2} & -1\end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix}3\\-2\end{pmatrix}$.

a) $\det(A - rI) = \det\begin{pmatrix}2-r & \frac{3}{2}\\ -\frac{3}{2} & -1-r\end{pmatrix} = (2-r)(-1-r) + \frac{9}{4} = r^2 - r + \frac{1}{4} = (r-\frac{1}{2})^2 \overset{\cdot}{=} 0$

so $r = \frac{1}{2}$ with algebraic multiplicity 2.

$(A - \frac{1}{2}I)\vec{\xi} = \vec{0}: \begin{pmatrix}3/2 & 3/2\\ -3/2 & -3/2\end{pmatrix} \to \begin{pmatrix}3/2 & 3/2\\ 0 & 0\end{pmatrix} \to \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ so $\xi_1 + \xi_2 = 0$, $\xi_1 = -\xi_2$

so $\vec{\xi} = \begin{pmatrix}-\xi_2\\ \xi_2\end{pmatrix} = \xi_2\begin{pmatrix}-1\\1\end{pmatrix}$ and $\vec{x}^{(1)} = \begin{pmatrix}-1\\1\end{pmatrix}e^{t/2}$.

Because the geometric multiplicity of $r = \frac{1}{2}$ is only 1, we find a second solution as $\vec{x}^{(2)} = \begin{pmatrix}-1\\1\end{pmatrix}te^{t/2} + \vec{\eta}\,e^{t/2}$ where $(A - \frac{1}{2})\vec{\eta} = \vec{\xi}$.

$\left(\begin{array}{cc|c}3/2 & 3/2 & -1\\ -3/2 & -3/2 & 1\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}3 & 3 & -2\\ 0 & 0 & 0\end{array}\right)$ so $3\eta_1 + 3\eta_2 = -2 \Rightarrow \eta_1 = \frac{-2-3\eta_2}{3}$.

With $\vec{\eta} = \begin{pmatrix}-\frac{2}{3} - \eta_2\\ \eta_2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}\eta_1 + \begin{pmatrix}-2/3\\0\end{pmatrix}$ with $\eta_1 = 0$, $\vec{\eta} = \begin{pmatrix}-2/3\\0\end{pmatrix}$

so $\vec{x}^{(2)} = \begin{pmatrix}-1\\1\end{pmatrix}te^{t/2} + \begin{pmatrix}-2/3\\0\end{pmatrix}e^{t/2} = \begin{pmatrix}-te^{t/2} - \frac{2}{3}e^{t/2}\\ te^{t/2}\end{pmatrix}$.

The general solution is

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$$
$$= c_1\begin{pmatrix}-1\\1\end{pmatrix}e^{t/2} + c_2\begin{pmatrix}-t - \frac{2}{3}\\ t\end{pmatrix}e^{t/2}.$$

To satisfy the initial condition:

$\vec{x}(0) = c_1\begin{pmatrix}-1\\1\end{pmatrix} + c_2\begin{pmatrix}-2/3\\0\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$ or $\begin{array}{l}-c_1 - \frac{2}{3}c_2 = 3\\ c_1 = -2 \Rightarrow c_1 = -2\end{array}$ $\quad c_2 = -\frac{3}{2}$

so $\vec{x} = -2\begin{pmatrix}-1\\1\end{pmatrix}e^{t/2} - \frac{3}{2}\begin{pmatrix}-t - 2/3\\ t\end{pmatrix}e^{t/2} = \begin{pmatrix}2 + \frac{3}{2}t + 1\\ -2 - \frac{3}{2}t\end{pmatrix}e^{t/2}$

$= \begin{pmatrix}3 + \frac{3}{2}t\\ -2 - \frac{3}{2}t\end{pmatrix}e^{t/2}$.