Solutions - HW #11

§5.2 #5. $(1-x)y'' + y = 0$, $x_0 = 0$

a) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$(1-x)y'' + y = y'' - xy'' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)n\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= (2\cdot 1\, a_2 + a_0)x^0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + a_n \right) x^n$$

$= 0$ provided $\boxed{\begin{array}{l} 2a_2 + a_0 = 0 \\ (n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + a_n = 0 \quad (n=1,2,\ldots) \end{array}}$

b) $2a_2 + a_0 = 0 \implies a_2 = -\frac{1}{2}a_0$

$n=1$: $3\cdot 2 a_3 - 2\cdot 1 a_2 + a_1 = 6a_3 - 2a_2 + a_1 = 6a_3 + a_0 + a_1 = 0 \implies a_3 = -\frac{1}{6}(a_0 + a_1)$

$n=2$: $4\cdot 3 a_4 - 3\cdot 2 a_3 + a_2 = \cancel{24}\,12 a_4 - 6\left(-\frac{1}{6}(a_0+a_1)\right) - \frac{1}{2}a_0 = \cancel{24}\,12 a_4 + \frac{1}{2}a_0 + a_1 = 0 \implies a_4 = -\frac{1}{24}a_0 - \frac{1}{12}a_1$

$n=3$: $5\cdot 4 a_5 - 4\cdot 3 a_4 + a_3 = 20a_5 - 12\left(-\frac{1}{24}a_0 - \frac{1}{12}a_1\right) - \frac{1}{6}(a_0+a_1) = 20a_5 + \frac{1}{3}a_0 + \frac{5}{6}a_1 = 0 \implies a_5 = -\frac{1}{60}a_0 - \frac{1}{24}a_1$

so $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^4 + a_5 x^5 + \ldots$

$$= a_0 + a_1 x + -\tfrac{1}{2}a_0 x^2 - \tfrac{1}{6}(a_0+a_1)x^3 + \left(-\tfrac{1}{24}a_0 - \tfrac{1}{12}a_1\right)x^4 + \left(-\tfrac{1}{60}a_0 - \tfrac{1}{24}a_1\right)x^5 + \ldots$$

$$= a_0\left(1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \ldots\right) + a_1\left(x - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 - \tfrac{1}{24}x^5 + \ldots\right)$$

$$\therefore \boxed{\begin{array}{l} y_1 = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} + \ldots \\ y_2 = x - \frac{1}{6}x^3 - \frac{x^4}{12} - \frac{x^5}{24} + \ldots \end{array}}$$

c) $W[y_1,y_2](0) = \det\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0)\end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} = 1 \neq 0.$

#8. $xy'' + y' + xy = 0$, $x_0 = 1$.

Because we have $x_0 = 1$, we must rewrite the DE with coefficients in powers of $x-1$ (not x). To do this we write $x = (x-1)+1$.

Thus, the DE we consider is

$$((x-1)+1)y'' + y' + ((x-1)+1)y = 0$$

or

$$(x-1)y'' + y'' + y' + (x-1)y + y = 0.$$

a) $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$.

$$(x-1)y'' + y'' + y' + (x-1)y + y = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$+ \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=1}^{\infty}(n+1)(n)\,a_{n+1}(x-1)^n + \sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}(x-1)^n + \sum_{n=0}^{\infty}(n+1)\,a_{n+1}(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^n$$

$$= (2\cdot 1\,a_2 + a_1 + a_0)(x-1)^0 + \sum_{n=1}^{\infty}\Big((n+2)(n+1)a_{n+2} + \big((n+1)n + (n+1)\big)a_{n+1} + a_n + a_{n-1}\Big)(x-1)^n$$

$$= (2a_2 + a_1 + a_0)(x-1)^0 + \sum_{n=1}^{\infty}\Big((n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1}\Big)(x-1)^n$$

$= 0$ provided

$$\boxed{\begin{array}{ll} 2a_2 + a_1 + a_0 = 0 & \\ (n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1} = 0 & (n = 1, 2, \dots) \end{array}}$$

b) $2a_2 + a_1 + a_0 = 0 \implies a_2 = -\frac{1}{2}(a_1 + a_0)$

$n=1$: $3\cdot 2a_3 + 2^2 a_2 + a_1 + a_0 = 6a_3 - 2a_0 - 2a_1 + a_1 + a_0 = 6a_3 - a_0 - a_1 = 0$

$$\implies a_3 = \frac{1}{6}(a_1 + a_0)$$

$n=2$: $4\cdot 3a_4 + 3^2 a_3 + a_2 + a_1 = 12a_4 + \frac{3}{2}(a_1 + a_0) - \frac{1}{2}(a_1 + a_0) + a_1 = 12a_4 + 2a_1 + a_0 = 0$

$$\implies a_4 = -\frac{1}{6}a_1 - \frac{1}{12}a_0$$

so $y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$

$= a_0 + a_1(x-1) - \frac{1}{2}(a_1 + a_0)(x-1)^2 + \frac{1}{6}(a_1 + a_0)(x-1)^3 + \left(-\frac{1}{6}a_1 - \frac{1}{12}a_0\right)(x-1)^4 + \dots$

$= a_0\left(1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \dots\right)$

$+ a_1\left((x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \dots\right)$

$$\therefore \boxed{\begin{array}{l} y_1 = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \dots \\ y_2 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \dots \end{array}}$$

$y_1' = -(x-1) + \frac{1}{2}(x-1)^2 + \dots$

$y_2' = -1 - (x-1) + \dots$

c) $W[y_1, y_2](1) = \det\begin{bmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$

§5.3 #4. $y'' + x^2 y' + (\sin x)y = 0$; $y(0) = a_0$, $y'(0) = a_1$

$$y''(0) + 0^2 y'(0) + (\sin 0)y(0) = y''(0) = 0 \implies \underline{y''(0) = 0.}$$

diff: $y''' + x^2 y'' + 2xy' + (\sin x)y' + (\cos x)y = 0$

$$y'''(0) + 0\,y''(0) + 0\,y'(0) + 0\,y'(0) + y(0) = y'''(0) + a_0 = 0 \implies \underline{y'''(0) = -a_0}$$

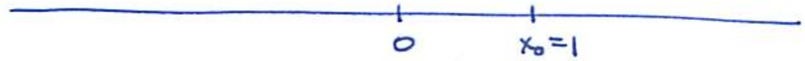
diff: $y^{(4)} + x^2 y''' + 2xy'' + 2xy'' + 2y' + (\sin x)y'' + (\cos x)y' + (\cos x)y' - (\sin x)y = 0$

$$y^{(4)}(0) + 0\,y'''(0) + 0\,y''(0) + 2y'(0) + 0\,y''(0) + 2y'(0) - 0\,y(0)$$

$$= y^{(4)}(0) + 2a_1 + 2a_1 = y^{(4)}(0) + 4a_1 = 0 \implies \underline{y^{(4)}(0) = -4a_1}$$

#8. $xy'' + y = 0$, $x_0 = 1$.

The coefficients are $P(x) = x$, $Q(x) = 0$, $R(x) = 1$.
All are continuous for all x.
$P(x) = 0 \iff x = 0$.



The shortest distance from x_0 to a discontinuity of P, Q, R or a zero of P is 1. So the radius of convergence is at least 1.

§5.4 #1. $x^2y'' + 4xy' + 2y = 0$

$y = x^r$: $x^2\, r(r-1)x^{r-2} + 4x\, r\, x^{r-1} + 2x^r = x^r\left(r(r-1) + 4r + 2\right)$
$= x^r\left(r^2 + 3r + 2\right) = x^r\,(r+2)(r+1)$.

For $x \neq 0$, we see that $r = -2$ & $r = -1$.

$\therefore\ y_1 = x^{-2}$ & $y_2 = x^{-1}$

The general solution is $y = c_1 x^{-2} + c_2 x^{-1}$.

#3. $x^2y'' - 3xy + 4y = 0$: $r(r-1) - 3r + 4 = r^2 - 4r + 4 = (r-2)^2 = 0$

$\therefore\ y_1 = x^2$ & $y_2 = x^2 \ln x$

The general solution is $y = c_1 x^2 + c_2 x^2 \ln x$.

#4. $x^2y'' + 3xy' + 5y = 0$: $r(r-1) + 3r + 5 = r^2 + 2r + 5 = (r+1)^2 + 4 = 0$

so $(r+1)^2 = -4$
$r + 1 = \pm 2i$
$r = -1 \pm 2i$

$\therefore\ y_1 = x^{-1}\cos(2\ln|x|)$ & $y_2 = x^{-1}\sin(2\ln|x|)$.

The general solution is $y = c_1 x^{-1}\cos(2\ln|x|) + c_2 x^{-1}\sin(2\ln|x|)$

#6. $(x-1)^2 y'' + 8(x-1)y' + 12y = 0$

$y = (x-1)^r$: $y' = r(x-1)^{r-1}$, $y'' = r(r-1)(x-1)^{r-2}$.

$(x-1)^2\, r(r-1)(x-1)^{r-2} + 8(x-1)\, r(x-1)^{r-1} + 12(x-1)^r$
$= r(r-1)(x-1)^r + 8r(x-1)^r + 12(x-1)^r$
$= \left(r(r-1) + 8r + 12\right)(x-1)^r = \left(r^2 + 7r + 12\right)(x-1)^r = (r+3)(r+4)(x-1)^r = 0$

so $r = -3$ or $r = -4$

$\therefore\ y_1 = (x-1)^{-3}$ & $y_2 = (x-1)^{-4}$

The general solution is $y = c_1(x-1)^{-3} + c_2(x-1)^{-4}$.