Exam 2 - Key

1. in standard form: $y'' + \frac{3}{t-4} y' + \frac{4}{t(t-4)} y = \frac{2}{t(t-4)}$, $y(-2) = 2, y'(-2) = 1$.

the coefs have discontinuities at $t = 0$ & $t = 4$.

since the IC are given at $t = -2$, the Ex. & Uniq. Thm. guarantees that a unique solution exists on $(-\infty, 0)$.

2. $x^2 y'' - x(x+2) y' + (x+2) y = 0$

(a) $y_1 = x$: $y_1' = 1$, $y_1'' = 0$

$x^2(0) - x(x+2)(1) + (x+2)x = 0$ ✓

$y_2 = xe^x$: $y_2' = xe^x + e^x = (x+1)e^x$, $y_2'' = (x+1)e^x + e^x = (x+2)e^x$

$$x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)xe^x = \left(x^2(x+2) - x(x+2)(x+1) + x(x+2)\right)e^x$$
$$= x(x+2)e^x\,(x - (x+1) + 1)$$
$$= 0. \checkmark$$

(b) To be a fundamental set of solutions on $x > 0$ requires $W[y_1, y_2] \neq 0$.

$$W[y_1, y_2] = \det \begin{bmatrix} x & xe^x \\ 1 & (x+1)e^x \end{bmatrix} = x(x+1)e^x - xe^x = x^2 e^x \neq 0 \text{ for } x > 0.$$

3. $y'' + 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = -2$.

Homog: $y = e^{rt}$, $r^2 + 2r + 5 = 0$, $r = \frac{1}{2}(-2 \pm \sqrt{4-20}) = \frac{1}{2}(-2 \pm 4i) = -1 \pm 2i$

$y_1 = e^{-t}\cos 2t \qquad y_2 = e^{-t}\sin 2t$.

$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$y' = -c_1 e^{-t}\cos 2t - 2c_1 e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t$

$y(0) = c_1 = 2$

$y'(0) = -c_1 + 2c_2 = -2 \quad$ so $\quad -2 + 2c_2 = -2$

$2c_2 = 0$

$c_2 = 0$

$\therefore y = 2e^{-t}\cos(2t)$.

4. Given $y_1 = t^{-1}$ is a solution to $t^2 y'' + 3ty' + y = 0$ we look for a 2nd solution

$$y_2 = v(t) y_1 = v(t) t^{-1}$$
$$y_2' = v' t^{-1} - v t^{-2}$$
$$y_2'' = v'' t^{-1} - v' t^{-2} - v' t^{-2} + 2v t^{-3} = v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}$$

$$t^2 y_2'' + 3t y_2' + y_2 = t^2 \left(v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}\right) + 3t\left(v' t^{-1} - v t^{-2}\right) + \left(v t^{-1}\right)$$
$$= t v'' + (-2+3) v' + (2-3+1) t^{-1} v$$
$$= t v'' + v' = 0$$

Let $u = v'$ so that $u' = v''$. Then $tu' + u = 0$.

This DE is separable: $t \frac{du}{dt} = -u$

$$\int \frac{du}{u} = -\int \frac{dt}{t}$$
$$\ln u = -\ln t = \ln(1/t) \Rightarrow u = \frac{1}{t}$$

So $v' = u = \frac{1}{t}$, which means $v = \int \frac{dt}{t} = \ln t$. and $y_2 = (\ln t) t^{-1} = \frac{\ln t}{t}$.

The general solution is $y = c_1 t^{-1} + c_2 \frac{\ln t}{t}$. $(t > 0)$.

5. $y'' + 2y' = 3 + 4\sin(2t)$

Homog: $y = e^{rt}$: $r^2 + 2r = 0$, $r(r+2) = 0 \Rightarrow r = 0$ or $r = -2$; $y_1 = e^{0t} = 1$, $y_2 = e^{-2t}$

$$y_h = c_1 (1) + c_2 e^{-2t}.$$

Partic: $g_1(t) = 3$: guess $y_{p_1} = At$ and multiply by t because A is in y_h.

$$y_{p_1}' = A$$
$$y_{p_1}'' = 0$$

$$0 + 2(A) = 3 \Rightarrow A = 3/2 \text{ and } y_{p_1} = \frac{3}{2} t.$$

$g_2(t) = 4\sin(2t)$: guess: $y_{p_2} = B\cos(2t) + C\sin(2t)$

$$y_{p_2}' = -2B\sin(2t) + 2C\cos(2t)$$
$$y_{p_2}'' = -4B\cos(2t) - 4C\sin(2t)$$

$$-4B\cos(2t) - 4C\sin(2t) + 2(-2B\sin(2t) + 2C\cos(2t))$$
$$= (-4B + 4C)\cos(2t) + (-4C - 4B)\sin(2t) = 4\sin(2t)$$

$-4B + 4C = 0 \Rightarrow B = C$

$-4C - 4B = 4$ $\quad -8C = 4 \Rightarrow C = -1/2$, $B = -1/2$ and $y_{p_2} = -\frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$

$\therefore$ General sol'n: $y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)$.

6. $y'' - 4y' + 4y = t^{-1}e^{2t}$

Homog: $y = e^{rt}$, $r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow r = 2, 2 \Rightarrow y_1 = e^{2t}$, $y_2 = te^{2t}$

Partic: we set $y_p = v_1y_1 + v_2y_2 = v_1 e^{2t} + v_2 te^{2t}$

where v_1, v_2 satisfy $\begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t}+e^{2t} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^{-1}e^{2t} \end{bmatrix}$.

$$e^{2t} v_1' + te^{2t} v_2' = 0$$
$$2e^{2t} v_1' + (2t+1)e^{2t} v_2' = t^{-1}e^{2t} \Longrightarrow$$

$$v_1' + t v_2' = 0 \quad (-2)$$
$$2v_1' + (2t+1)v_2' = t^{-1} \quad +$$

$$v_2' = t^{-1}$$
$$v_1' = -tv_2' = -t(t^{-1}) = -1.$$

so $v_2 = \int t^{-1}\,dt = \ln t$

$v_1 = \int -1\,dt = -t$

$\therefore\ \underline{y_p = -t e^{2t} + (\ln t) t e^{2t}}$.

Note: Because $-te^{2t}$ is in the homogeneous solution it's a little simpler to state that a particular solution is $(\ln t) t e^{2t}$

But, this is not the particular solution found by variation of parameters.