MATH 520 (Section 001)
Prof. Meade

Exam 1
September 20, 2011

University of South Carolina
Fall 2011

Name: Key
SS # (last 4 digits): ____________

Instructions:

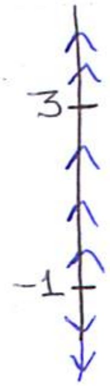
1. There are a total of 5 problems on 3 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be clearly labeled and written legibly on additional sheets of paper (that I will provide). Be sure each sheet contains your name and the work for each question is clearly labeled.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	10	
2	20	
3	36	
4	20	
5	14	
Total	100	

Good Luck!

Solutions: Exam #1

#1.



$$\lim_{t\to\infty} y(t) = \begin{cases} +\infty & \text{if } y_0 > 3 \\ 3 & \text{if } -1 < y_0 \le 3 \\ -1 & \text{if } y_0 = -1 \\ -\infty & \text{if } y_0 < -1 \end{cases}$$

#2. A: vii   B: viii   C: v   D: iv.

#3. a) $ty' - y = t^2 e^{-t}$, $t > 0$.

$$y' - \tfrac{1}{t} y = t e^{-t}$$

$$\frac{d}{dt}\left(\tfrac{1}{t} y\right) = \tfrac{1}{t} t e^{-t} = e^{-t}$$

$$\tfrac{1}{t} y = -e^{-t} + C$$

$$\underline{y = -t e^{-t} + Ct}$$

Integrating factor:

$$\frac{d}{dt}(\mu y) = \mu y' - \frac{\mu}{t} y$$

$$\cancel{\mu y'} + \mu' y = \cancel{\mu y'} - \frac{\mu}{t} y$$

$$\mu' y = -\frac{\mu}{t} y$$

$$\mu' = -\frac{\mu}{t} \quad \text{(separable)}$$

$$\frac{d\mu}{\mu} = -\frac{dt}{t}$$

$$\ln \mu = -\ln t = \ln(t^{-1})$$

$$\mu = t^{-1} = \frac{1}{t}.$$

b) $\underbrace{(e^x \sin y + 3y)}_{M} + \underbrace{(3x + y^2 + e^x \cos y)}_{N} y' = 0.$

$$\frac{\partial M}{\partial y} = e^x \cos y + 3 \qquad \frac{\partial N}{\partial x} = 3 + e^x \cos y \implies \text{exact!}$$

Solution is $\psi(x,y) = C$ where

$$\frac{\partial \psi}{\partial x} = M = e^x \sin y + 3y \implies \psi = e^x \sin y + 3xy + g(y)$$

$$\frac{\partial \psi}{\partial y} = N = 3x + y^2 + e^x \cos y \qquad \frac{\partial \psi}{\partial y} = e^x \cos y + 3x + g'(y) \implies g'(y) = y^2$$

$$g(y) = \tfrac{1}{3} y^3$$

$$\therefore \psi(x,y) = e^x \sin y + 3xy + \tfrac{1}{3} y^3 = C$$

c) $y' = xy^2(1-x^2)^{-1/2}$

$$\int \frac{dy}{y^2} = -\frac{1}{2}\int \frac{-2x}{(1-x^2)^{1/2}}\,dx \qquad u = 1-x^2, \quad du = -2x\,dx$$

$$-y^{-1} = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = -(1-x^2)^{1/2} + C.$$

$$\frac{1}{y} = (1-x^2)^{1/2} + C$$

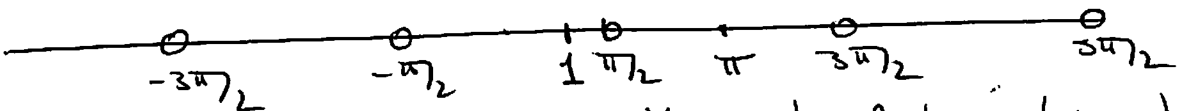
$$y = \frac{1}{(1-x^2)^{1/2} + C}.$$

#4a) $(\cos t)\,y' + (\sin t)\,y = \cos t \sin t$: 1st order linear but not in standard form

$$y' + (\tan t)\,y = \sin t$$ in standard form

$\sin t$: continuous for all t

$\tan t$: continuous for all t except odd multiples of $\pi/2$.



The interval that contains the initial time $(t=\pi)$ is $(+\pi/2,\ 3\pi/2)$.

b) $$y' = \frac{\ln|ty|}{1-t^2-y^2} = f(t,y)$$ 1st order, nonlinear

Need to look at continuity of $f(t,y)$ & $\frac{\partial f}{\partial y}(t,y)$

$f(t,y) = \frac{\ln|ty|}{1-t^2-y^2}$ is continuous when $|ty| \neq 0 \iff ty \neq 0 \iff t \neq 0 \;\&\; y \neq 0.$

and $1-t^2-y^2 \neq 0 \iff t^2+y^2 = 1.$

$$\frac{\partial f}{\partial y} = \frac{(1-t^2-y^2)\frac{1}{\cancel{t}y}\cancel{t} - (-2y)\ln|ty|}{(1-t^2-y^2)^2}$$

$$= \frac{1}{y(1-t^2-y^2)} + \frac{2y\ln|ty|}{(1-t^2-y^2)^2}$$ is continuous on the same set

Any initial condition $y(t_0)=y_0$ can be given except those with $t_0^2+y_0^2 = 1$ or $t_0 = 0$ or $y_0 = 0$.

5. $$\underbrace{(x\sec y + 3x^3\tan y)}_{M} + \underbrace{(xy^2\sec y + x^4)}_{N}\, y' = 0.$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= x\sec y\tan y + 3x^3\sec^2 y \\ \frac{\partial N}{\partial x} &= y^2\sec y + 4x^3 \end{aligned}\right\}\ \text{not exact.}$$

Multiply by $\mu = \frac{\cos y}{x}$:

$$\underbrace{(1 + 3x^2\sin y)}_{\mu M} + \underbrace{(y^2 + x^3\cos y)}_{\mu N}\, y' = 0.$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}(\mu M) &= 3x^2\cos y \\ \frac{\partial}{\partial x}(\mu N) &= 3x^2\cos y \end{aligned}\right\}\ \text{exact!}$$

Solution will be $\psi(x,y) = C$ where

$$\frac{\partial \psi}{\partial x} = \mu M = 1 + 3x^2\sin y \implies \psi = x + x^3\sin y + g(y).$$

$$\frac{\partial \psi}{\partial y} = \mu N = y^2 + x^3\cos y \qquad \frac{\partial \psi}{\partial y} = x^3\cos y + g'(y)$$

$$\implies g'(y) = y^2$$

$$\implies g(y) = \tfrac{1}{3}y^3$$

so $\psi(x,y) = \underline{x + x^3\sin y + \tfrac{1}{3}y^3 = C}$