MATH 520 (Section 501)
Prof. Meade

University of South Carolina
Fall 2006

Exam 2
October 18, 2006

Name: Key
SS # (last 4 digits): ________

Instructions:

1. There are a total of problems on 5 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) <u>and</u> you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	24	
2	20	
3	28	
4	28	
Total	100	

I hope you have a great Fall Break!

1. (24 points) [12 points each]

(a) Find the general solution to $y'' - 2y' + 3y = 0$.

$y = e^{rt}$: $P(r) = r^2 - 2r + 3 = 0.$

$$r = \frac{1}{2}\left(2 \pm \sqrt{4-12}\right)$$
$$= \frac{1}{2}\left(2 \pm \sqrt{-8}\right)$$
$$= \frac{1}{2}\left(2 \pm 2\sqrt{2}\,i\right)$$
$$= 1 \pm \sqrt{2}\,i.$$

$y_1 = e^t \cos(\sqrt{2}\,t)$

$y_2 = e^t \sin(\sqrt{2}\,t)$

$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^t \cos(\sqrt{2}\,t) + c_2 e^t \sin(\sqrt{2}\,t)$

(b) Find the general solution to $y''' - 3y'' + 3y' - y = 0$.

$y = e^{rt}$: $P(r) = r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

$r = 1$ (mult. 3).

$y_1 = e^t$

$y_2 = te^t$

$y_3 = t^2 e^t$

$y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3$

$= c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$

2. (20 points) [5 points each — yes, I know $5 \times 5 = 25$; there are 5 extra points in this problem]

(a) An object with mass 10 kg oscillates on the end of a spring with period 2 s. Explain why the spring constant is $k = 10\pi^2$.

NOTE: *Assume there is no damping.*

The differential equation is $my'' + ky = 0$.
Because the period is 2, we know $y = c_1 \cos(\pi t) + c_2 \sin(\pi t)$; i.e. $r = \pm\pi i$ are the roots of $P(r) = mr^2 + k$. Thus: $m(\pi i)^2 + k = 0$

$k = -m(-\pi^2) = 10\pi^2$.

(b) Continuing (a), for what value of the damping coefficient, b, is the motion critically damped?

$my'' + by' + ky = 0.$

$10y'' + by' + 10\pi^2 y = 0.$

$y = e^{rt}$: $\quad P(r) = 10r^2 + br + 10\pi^2 = 0.$

$r = \frac{1}{2 \cdot 10}\left(-b \pm \sqrt{b^2 - 400\pi^2}\right)$

Critically damped when $b^2 - 400\pi^2 = 0$.

$b = 400\pi^2$ so, $\underline{b = 20\pi}$

(c) Find the Wronskian of $y_1 = \ln(x)$ and $y_2 = x\ln(x)$.

$$W[y_1, y_2] = \det\begin{bmatrix} \ln(x) & x\ln(x) \\ \frac{1}{x} & 1 \cdot \ln(x) + x \cdot \frac{1}{x} \end{bmatrix} = \det\begin{bmatrix} \ln(x) & x\ln(x) \\ \frac{1}{x} & \ln(x) + 1 \end{bmatrix}$$

$$= \ln(x)\left(\ln(x) + 1\right) - \frac{1}{x} x\ln(x) = (\ln(x))^2 + \ln(x) - \ln(x)$$

$$= (\ln(x))^2.$$

(d) Explain why the two functions in (c) could *not* be solutions to a homogeneous second-order linear differential equation on the interval $(0, \infty)$.

Notice that $W[y_1, y_2](1) = (\ln(1))^2 = 0$, but that $W[y_1, y_2](x) \neq 0$ for all other $x > 0$. By Abel's Theorem, the Wronskian of 2 lin. indep sol'ns must be either always zero or never zero.

(e) Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions to the nth-order linear differential equation

$$y^{(n)} + p_2(x)y^{(n-2)} + p_3(x)y^{(n-3)} + \cdots + p_n(x)y = 0.$$

Use Abel's Theorem to show that the Wronskian, $W[y_1, y_2, \ldots, y_n]$, is a constant.

NOTE: *Note that there is no $y^{(n-1)}$ term.*

$W' = -p_1 W = 0 \implies W = \text{constant}$
(by Abel's Thm.)

3. (28 points) Consider the differential equation

$$\sin(x)y'' + (1+\sin(x))y' + y = 0.$$

(a) [10 points] Show that any solution of the form $y = e^{rx}$ must satisfy.

$$(r^2+r)\sin(x) + (r+1) = 0.$$

$y = e^{rx}$: $y' = re^{rx}$, $y'' = r^2e^{rx}$

$0 = \sin(x)\, y'' + (1+\sin(x))y' + y.$

$= \sin(x)\, r^2e^{rx} + (1+\sin(x))re^{rx} + e^{rx}$

$= e^{rx}\left((r^2+r)\sin(x) + (r+1)\right)$ so $(r^2+r)\sin(x) + r+1 = 0.$

(b) [10 points] Because the above condition must be satisfied for all x, the only solution is $r=-1$. That is, $y_1 = e^{-x}$. A second solution will be sought in the form $y_2(x) = v(x)y_1(x)$. Find the first-order differential satisfied by $u = v'$.

$y_2 = v\,e^{-x}$: $y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$

$0 = \sin(x)(v''-2v'+v)e^{-x} + (1+\sin(x))(v'e^{-x} - ve^{-x}) + ve^{-x}$

$= \sin(x)e^{-x}v'' + (-2\sin(x) + 1 + \sin(x))e^{-x}v' + (\sin(x) - (1+\sin(x)) + 1)e^{-x}v$

so

$0 = (\sin(x)v'' + (1-\sin(x))v' + 0v)e^{-x}$

Because $e^{-x} \neq 0$ for all (real) x:

$\sin(x)v'' + (1-\sin(x))v' = 0.$

Let $u = v'$: $\underline{\sin(x)u' + (1-\sin(x))u = 0}$

(c) [4 points] With initial conditions $y(\pi/4) = 1$, $y'(\pi/4) = -1$, what is the largest interval on which a unique solution is guaranteed to exist by the Existence and Uniqueness Theorem (Theorem 2.4.2)?

$y'' + \frac{1+\sin(x)}{\sin(x)}y' + \frac{1}{\sin(x)}y = 0.$

Because $\sin(x) \neq 0$ for $x \in (0, \pi/2)$, the E&U Theorem tells us the solution must exist & be unique for (at least) $0 < x < \pi/2$.

(d) [4 points] With initial conditions $y(0) = 1$, $y'(0) = -1$, what is the largest interval on which a unique solution is guaranteed to exist by the Existence and Uniqueness Theorem (Theorem 2.4.2)?

Because $\sin(0) = 0$ the coef. of y' is not continuous at $x=0$, the E&U Theorem has nothing to say about this problem.

4. (28 points) Consider the Cauchy–Euler equation

$$x^2y'' - 2y = 0.$$

(a) [12 points] Find the general solution.

$x = e^t$: $y'' - y' - 2y = 0$.

$y = e^{rt}$ $P(r) = r^2 - r - 2$

$= (r-2)(r+1)$

$r = 2, r = -1$

$y_1 = e^{2t}$

$y_2 = e^{-t}$

$y = c_1 e^{2t} + c_2 e^{-t}$

$= c_1 x^2 + c_2 x^{-1}$

(b) [8 points] Find the solution that satisfies the initial conditions

$$y(1) = \alpha \qquad y'(1) = 2.$$

$y = c_1 x^2 + c_2 x^{-1}$

$y' = 2c_1 x - c_2 x^{-2}$

$y(1) = c_1 + c_2 = \alpha$

$y'(1) = 2c_1 - c_2 = 2$

$3c_1 = 2 + \alpha$ $\quad c_1 = \frac{2+\alpha}{3}$

$c_2 = \alpha - c_1 = \alpha - \frac{2+\alpha}{3} = \frac{3\alpha - 2 - \alpha}{3}$

$= \frac{2\alpha - 2}{3}$.

so $y = \frac{2+\alpha}{3} x^2 + \frac{2\alpha - 2}{3} x^{-1}$.

(c) [4 points] For what value(s) of α does the solution approach 0 as $x \to \infty$?

$\lim_{x\to\infty} y(x) = 0$ only when $\frac{2+\alpha}{3} = 0$, i.e. $\underline{\alpha = -2}$

(d) [4 points] For what value(s) of α is the solution bounded as $x \to 0^+$?

$\lim_{x\to 0^+} y(x)$ exists only when $\frac{2\alpha-2}{3} = 0$, i.e. $\underline{\alpha = 1}$