

Exam 3
April 21, 2004

Name: _____
SS #: _____
Key

Instructions:

1. There are a total of 5 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. You must show all of your work to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
4. The table of distributions and tables for the χ^2 and normal distributions will be handed out separately.

Problem	Points	Score
1	15	
2	20	
3	15	
4	35	
5	15	
Extra Credit	10	
Total	100	

Good Luck!!

1. (15 points) [5 points each] The moment generating function of X is $M(t) = (1-2t)^{-8}$, $t < 1/2$.
 Find

(a) $E[X]$ \underline{X} is $\chi^2(16)$, or gamma with $\alpha=8, \theta=2$.

$$E[\underline{X}] = \mu = r = \alpha\theta = 16$$

(b) $\text{Var}(X)$.

$$\text{Var}(\underline{X}) = \sigma^2 = 2r = \alpha\theta^2 = 32$$

$$(c) P(6.908 < X < 9.312) = P(\underline{X} \leq 9.312) - P(\underline{X} \leq 6.908)$$

$$\begin{aligned} * \text{ really need to use } \chi^2(16) &= 0.100 - 0.025 \\ \text{for this part; and use} &= 0.075. \end{aligned}$$

Table IV.

2. (20 points) [5 points each] Let X have an exponential distribution with mean 10.

(a) Find $P(10 < X < 20)$. \underline{X} is exponential with $\mu = \theta = 10$; $f(x) = \frac{1}{10} e^{-x/10}$

$$\begin{aligned} P(10 < X < 20) &= \int_{10}^{20} f(x) dx \\ &= \int_{10}^{20} \frac{1}{10} e^{-x/10} dx \\ &= -e^{-x/10} \Big|_{10}^{20} = -e^{-2} + e^{-1} \approx 0.2325 \end{aligned}$$

(b) Find $P(X > 10)$.

$$P(\underline{X} > 10) = \int_{10}^{\infty} f(x) dx = \int_{10}^{\infty} \frac{1}{10} e^{-x/10} dx = -e^{-x/10} \Big|_{10}^{\infty} = -0 + e^{-1} \approx 0.3679$$

$$\begin{aligned} (c) \text{ Find } P(X < 20 | X > 10) &= \frac{P(\underline{X} < 20 \text{ and } X > 10)}{P(\underline{X} > 10)} \\ &= \frac{P(10 < \underline{X} < 20)}{P(\underline{X} > 10)} \\ &= \frac{e^{-1} - e^{-2}}{e^{-1}} = 1 - e^{-1} \approx 0.6321 \end{aligned}$$

(d) What is the median value of X ?

For one point I will give you the definition of median.

The median value of \underline{X} is the number m such that $P(\underline{X} \leq m) = 1/2$.

$$\text{So: } \frac{1}{2} = P(\underline{X} \leq m) = \int_{-\infty}^m \frac{1}{10} e^{-x/10} dx = -e^{-x/10} \Big|_{-\infty}^m = -e^{-m/10} + 1$$

$$\text{Thus, } \frac{1}{2} = 1 - e^{-m/10}$$

$$e^{-m/10} = \frac{1}{2}$$

$$-m/10 = \ln(\frac{1}{2}) = -\ln 2 \implies m = 10 \ln 2 \approx 6.9314$$

3. (15 points) [5 points each] A company manufactures bags of peanuts whose weight is normally distributed with a mean of 40 grams and a variance of 25. Let X denote the weight of a particular bag of peanuts.

(a) What is the moment generating function of X ? $\underline{X} \sim N(40, 25)$

$$M(t) = e^{\mu t + \frac{\sigma^2}{2}t^2} = e^{40t + \frac{25}{2}t^2}$$

$$(b) \text{ Find } P(37 \leq X \leq 46) = P\left(\frac{37-40}{5} \leq \frac{X-40}{5} \leq \frac{46-40}{5}\right)$$

$$\begin{aligned} Z &= \frac{X-40}{5} \sim N(0,1) = P(-0.6 \leq Z \leq 1.2) \\ &= \Phi(1.2) - \Phi(-0.6) \\ &= \Phi(1.2) - (1 - \Phi(0.6)) \\ &\approx 0.8849 - (1 - 0.7257) \\ &= 0.6106 \end{aligned}$$

$$(c) \text{ Find a number } b \text{ such that } P\left(\frac{|X-40|}{5} \leq b\right) = 0.34.$$

again, use $Z = \frac{X-40}{5} \sim N(0,1)$ to write

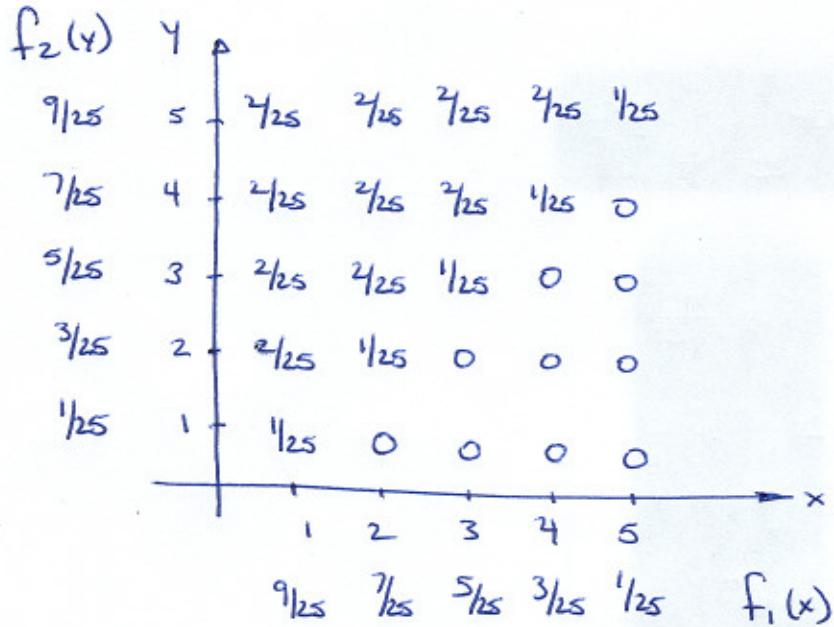
$$\begin{aligned} 0.34 &= P\left(\frac{|X-40|}{5} \leq b\right) \\ &= P(|Z| \leq b) \\ &= P(-b \leq Z \leq b) \\ &= \Phi(b) - \Phi(-b) \\ &= \Phi(b) - (1 - \Phi(b)) \\ &= 2\Phi(b) - 1 \end{aligned}$$

$$\text{so } 2\Phi(b) = 1.34 \text{ and so } \Phi(b) = 0.67$$

$$\text{From Table IVa: } b = 0.44.$$

4. (35 points) Roll a pair of five-sided dice. Let X equal the smaller outcome and let Y equal the larger outcome.

- (a) [18 points] Display the joint p.m.f. on a graph along with the marginal probabilities.



x	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

y	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

$$E[X^2]$$

- (b) [12 points] Find μ_X , σ_X^2 and $E[XY]$.

$$\mu_X = E[X] = 1 \cdot \frac{9}{25} + 2 \cdot \frac{7}{25} + 3 \cdot \frac{5}{25} + 4 \cdot \frac{3}{25} + 5 \cdot \frac{1}{25} = \frac{55}{25} = \frac{11}{5} = 2.2$$

$$E[X^2] = 1^2 \cdot \frac{9}{25} + 2^2 \cdot \frac{7}{25} + 3^2 \cdot \frac{5}{25} + 4^2 \cdot \frac{3}{25} + 5^2 \cdot \frac{1}{25} = \frac{395}{25} = \frac{79}{5} = 15.8$$

$$\begin{aligned}
 E[XY] &= 1 \cdot 1 \cdot \frac{9}{25} + 1 \cdot 2 \cdot \frac{7}{25} + 1 \cdot 3 \cdot \frac{5}{25} + 1 \cdot 4 \cdot \frac{3}{25} + 1 \cdot 5 \cdot \frac{1}{25} && (\frac{29}{25}) \\
 &+ 2 \cdot 1 \cdot \frac{7}{25} + 2 \cdot 2 \cdot \frac{2}{25} + 2 \cdot 3 \cdot \frac{4}{25} + 2 \cdot 4 \cdot \frac{1}{25} + 2 \cdot 5 \cdot \frac{0}{25} && (\frac{52}{25}) \\
 &+ 3 \cdot 1 \cdot \frac{5}{25} + 3 \cdot 2 \cdot \frac{4}{25} + 3 \cdot 3 \cdot \frac{1}{25} + 3 \cdot 4 \cdot \frac{0}{25} && (\frac{63}{25}) \\
 &+ 4 \cdot 1 \cdot \frac{3}{25} + 4 \cdot 2 \cdot \frac{1}{25} + 4 \cdot 3 \cdot 0 && (\frac{56}{25}) \\
 &+ 5 \cdot 1 \cdot 0 && (\frac{25}{25}) \\
 \\
 &= \frac{225}{25} = 9.
 \end{aligned}$$

- (c) [5 points] Are X and Y independent? (Explain.)

No. For example, $f(1,1) = \frac{1}{25} \neq f_1(1)f_2(1)$.

5. (15 points) [5 points each] Let X and Y have a joint p.m.f. given by

$$f(x, y) = \frac{xy}{60}, \quad x = 1, 2, 3, \quad y = 1, 2, 3, 4.$$

(a) Find the marginal p.m.f. for X .

$$\begin{aligned} f_1(x) &= \sum_{y=1}^4 f(x, y) \\ &= f(x, 1) + f(x, 2) + f(x, 3) + f(x, 4) \\ &= \frac{x}{60} + \frac{2x}{60} + \frac{3x}{60} + \frac{4x}{60} \\ &= \frac{10x}{60} \\ &= \frac{x}{6} \quad \text{for } x=1,2,3. \quad (0 \text{ otherwise}) \end{aligned}$$

(b) Find the conditional p.m.f. of Y , given $X = x$.

$$\begin{aligned} h(y|x) &= P(Y=y | X=x) \\ &= \frac{P(Y=y, X=x)}{P(X=x)} \\ &= \frac{f(x,y)}{f_1(x)} \\ &= \frac{xy/60}{x/60} \\ &= \frac{y}{10} \quad \text{for } y=1,2,3,4 \quad (0 \text{ otherwise}) \end{aligned}$$

(c) Find $P(2 \leq Y \leq 4 | X = 2)$.

$$\begin{aligned} &= h(2|2) + h(3|2) + h(4|2) \\ &= \frac{2}{10} + \frac{3}{10} + \frac{4}{10} \\ &= \frac{9}{10}. \end{aligned}$$

Extra Credit (10 points) [1 point each] Each of the following functions appeared as a p.m.f. on at least one submitted question. Identify which of these functions is a p.m.f. (Briefly explain each answer.)

(a) $f(x, y) = \frac{1}{4}$, $(x, y) \in \{(0, 0), (1, 1), (1, -1), (2, 0)\}$

1) $0 \leq \frac{1}{4} \leq 1$

2) $\sum_{(x,y) \in S} f(x, y) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1 \therefore \text{Yes, this is a p.m.f.}$

(b) $f(x, y) = \frac{1}{4}x^2y$, $x=0, 1, 2, 3$, $y=0, 1, 2, 3, 4$

For $x=2, y=2 : f(2, 2) = 2 > 1 \therefore \text{No, not a p.m.f.}$

To be a p.d.f., recall:

1) $0 \leq f(x, y) \leq 1$

2) $\sum_{(x,y) \in S} f(x, y) = 1$

(c) $f(x, y) = \frac{3}{4}x^2y$, $x=1, 2$, $y=1, 2, 3, 4$

(same logic as (b))

No, not a p.d.f.

(d) $f(x, y) = \frac{x+y}{32}$, $x=1, 2$, $y=1, 2, 3, 4$

1) $0 \leq f(x, y) \leq \frac{2+4}{32} = \frac{3}{16} < 1$

2) $\sum_{y=1}^4 \sum_{x=1}^2 f(x, y) = \sum_{y=1}^4 \frac{2y+3}{32} = \frac{5+7+9+11}{32} = 1 \therefore \text{Yes, this is a p.d.f.}$

(e) $f(x, y) = \frac{x+y}{20}$, $x=1, 2$, $y=1, 2, 3, 4$

$\sum_{y=1}^4 \sum_{x=1}^2 f(x, y) = \frac{32}{20} > 1$

$\therefore \text{No, not a p.d.f.}$

(f) $f(x, y) = \frac{x+y}{16}$, $x=1, 2$, $y=1, 2, 3, 4$

$\sum_{y=1}^4 \sum_{x=1}^2 f(x, y) = \frac{32}{16} > 1$

$\therefore \text{No, not a p.d.f.}$

(g) $f(x, y) = \frac{x+y}{39}$, $x=1, 2, 3$, $y=1, 2, 3$

$\sum_{y=1}^3 \sum_{x=1}^3 f(x, y) = \frac{36}{39} \neq 1$

$\therefore \text{No, not a p.d.f.}$

(h) $f(x, y) = \frac{x+y}{21}$, $x=1, 2, 3$, $y=1, 2$

$\sum_{y=1}^2 \sum_{x=1}^3 f(x, y) = \frac{21}{21} = 1$

$\therefore \text{Yes, this is a p.d.f.}$

(i) $f(x, y) = \frac{2x+y}{18}$, $x=1, 2$, $y=1, 2$

$\sum_{y=1}^2 \sum_{x=1}^2 \frac{2x+y}{18} = \frac{18}{18} = 1$

$\therefore \text{Yes, this is a p.d.f.}$

(j) $f(x, y) = \frac{x+2y}{20}$, $x=1, 2$, $y=1, 2$

$\sum_{y=1}^2 \sum_{x=1}^2 \frac{x+2y}{20} = \frac{18}{20} \neq 1$

$\therefore \text{No, not a p.d.f.}$