MATH/STAT 511
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University of South Carolina
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Exam 2
March 26, 2004

Name: Key
SS #:

Instructions:

1. There are a total of 7 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. You must show all of your work to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
4. The last page is a table of distributions. Tear off this page.

Problem	Points	Score
1	8	
2	8	
3	12	
4	16	
5	16	
6	20	
7	20	
Total	100	

Good Luck!!

1. (8 points) [4 points each] Let X be a discrete random variable with space $S = \{-1, 0, 2\}$ and probability mass function with $f(-1) = 0.6$ and $f(0) = 0.3$.

   (a) What are $f(2)$ and $f(-2)$?

   $f(-2) = 0$ because $-2 \notin S$

   $$\begin{aligned} f(2) &= 1 - (f(-1) + f(0)) \\ &= 1 - 0.9 \\ &= 0.1 \end{aligned}$$

   (b) Find $\mathrm{E}[X^2]$.

   $$\begin{aligned} E[X^2] = \sum_{x \in S} x^2 f(x) &= (-1)^2 (0.6) + 0^2 (0.3) + (2)^2 (0.1) \\ &= 0.6 + 0.4 \\ &= 1. \end{aligned}$$

2. (8 points) [4 points each] Let Y be a random variable with $\mathrm{E}[Y^2] = 15$.

   (a) If $\mathrm{E}[Y] = 3$, what is σ?

   $$\begin{aligned} \sigma^2 &= E[Y^2] - E[Y]^2 \\ &= 15 - 3^2 \\ &= 6. \end{aligned}$$

   (b) Use $\mathrm{Var}[Y]$ to explain why it is not possible to have $\mathrm{E}[Y] = 4$.

   $\mathrm{Var}[Y] = E[(Y-\mu)^2] \geq 0.$

   but, if $E[Y] = 4$

   then $E[Y^2] - E[Y]^2 = 15 - 4^2 = -1 < 0.$

   Since this is not possible, it must be that $E[Y] \neq 4.$

   (In fact, $|E[Y]| < \sqrt{15}$.)

3. (12 points) [4 points each] Let X be a random variable with moment generating function

$$M(t) = e^{6e^t - c}.$$

(a) Explain why you know $c = 6$.

$M(0) = 1$ for every m.g.f.

Here, $M(0) = e^{6e^0 - c} = e^{6-c} = 1 \iff 6 - c = 0$

$\iff c = 6.$

(b) What is the name of the distribution of X?

Poisson with parameter $\lambda = 6$.

(c) What is the mean of X.

$\mu = \lambda = 6.$

4. (16 points) [8 points each] For each of the following moment generating functions identify the probability mass function and the space of the distribution.

(a) $M(t) = 0.3e^{-2t} + 0.2 + 0.2e^t + 0.1e^{4t} + 0.2e^{5t}$

$$f(x) = \begin{cases} 0.3 & x = -2 \\ 0.2 & x = 0 \\ 0.2 & x = 1 \\ 0.1 & x = 4 \\ 0.2 & x = 5 \end{cases} \qquad S = \{-2, 0, 1, 4, 5\}.$$

(b) $M(t) = \frac{(3+2e^t)^4}{625} = \frac{(3+2e^t)^4}{5^4} = \left(\frac{3}{5} + \frac{2}{5}e^t\right)^4.$ NOTE: $625 = 5^4$.

binomial w/ $n = 4$, $p = 2/5$.

$$f(x) = \binom{4}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{4-x}.$$

$S = \{0, 1, 2, 3, 4\}.$

5. (16 points) [4 points each] Treatment for a disease has a 90% success rate. The treatment is given (independently) to a group of 10 patients. Let X denote the number of patients for whom the treatment is successful.

(a) What is the name of the distribution of X?

Be sure to include values for all parameters.

$b(10, 0.9)$

(b) What is $P(X = 5)$?

$$P(X=5) = \binom{10}{5}(0.9)^5(0.1)^5$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!}(0.9)^5(0.1)^5$$

(c) What is the mean of X?

$$\mu = np = 10(0.9) = 9$$

(d) What is $\mathrm{E}[X^2]$?

$$\sigma^2 = E[X^2] - E[X]^2$$
$$\text{so } E[X^2] = \sigma^2 + \mu^2 = np(1-p) + (np)^2$$
$$= 10(0.9)(0.1) + 9^2$$
$$= 0.9 + 81$$
$$= 81.9$$

6. (20 points) [5 points each] A biased six-sided die results in a 2 on $\frac{1}{10}$ of the rolls. Let X denote the number of 2's in 1000 rolls of the die.

(a) What is the exact probability mass function for X?

binomial, $n = 1000, p = 0.1$

$$f(x) = \binom{1000}{x} 0.1^x \, 0.9^{1000-x} \quad , \quad x = 0, 1, 2, \ldots, 1000$$

(b) Find $P(X = 3)$.

Do not attempt to find a numerical value for this answer. Leave your answer in terms of factorials and powers.

$$P(X=3) = f(3) = \binom{1000}{3} 0.1^3 \, 0.9^{997}$$

$$= \frac{1000!}{3! \, 997!} 0.1^3 \, 0.9^{997}$$

$$= \frac{1000 \cdot 999 \cdot 998}{6} 0.1^3 \, 0.9^{997}$$

$$= 0.0001488$$

(c) What is an approximate probability mass function for X?

Poisson with $\lambda = np = 100$.

$$f(x) = \frac{100^x e^{-100}}{x!}$$

(d) Use your answer in (c) to approximate $P(3 \leq X \leq 7)$?

$$P(3 \leq X \leq 7) = P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)$$

$$= \frac{100^3 e^{-100}}{3!} + \frac{100^4 e^{-100}}{4!} + \frac{100^5 e^{-100}}{5!} + \frac{100^6 e^{-100}}{6!} + \frac{100^7 e^{-100}}{7!}$$

$$= \frac{100^3}{3!} e^{-100} \left(1 + \frac{100}{4} + \frac{100^2}{4 \cdot 5} + \frac{100^3}{4 \cdot 5 \cdot 6} + \frac{100^4}{4 \cdot 5 \cdot 6 \cdot 7} \right)$$

$$= 7.93 \times 10^{-34}$$

Note: If you use the binomial distribution from (a), $P(3 \leq X \leq 7) = 7.575 \times 10^{-36}$

7. (20 points) [4 points each] Let X be a continuous random variable.

(a) Find c such that $f(x) = \frac{1}{4}x^3$, $0 < x < c$, is a probability density function for X.

need to have $\int_0^c f(x)dx = 1$:

$\int_0^c \frac{1}{4}x^3 dx = \frac{1}{16}x^4 \Big|_0^c = \frac{c^4}{16} - 0 = 1 \iff c^4 = 16$

$\iff c = 2$ (not $c = -2$, because $c > 0$)

(b) Find the cumulative distribution function for X.

Be sure your cdf is defined for all real numbers.

For $x \in (0,2)$: $F(x) = \int_{-\infty}^x f(t)dt = \int_0^x \frac{1}{4}t^3 dt = \frac{1}{16}t^4\Big|_0^x = \frac{x^4}{16}$.

For $x \le 0$: $F(x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^x 0\, dx = 0$

For $x \geq 2$: $F(x) = \int_{-\infty}^x f(t)dt = \int_0^2 f(t)dt = 1$

so $F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^4}{16} & 0 < x < 2 \\ 1 & x \geq 2. \end{cases}$

(c) Find $P(X = \frac{1}{4})$.

Because X has a continuous distribution,

$P(X = 1/4) = 0$.

(d) Find $P(\frac{1}{2} < X < 2)$.

$P(\frac{1}{2} < X < 2) = \int_{1/2}^2 f(x)dx = \int_{1/2}^2 \frac{1}{4}x^3 dx = \frac{1}{16}x^4\Big|_{1/2}^2 = \frac{16}{16} - \frac{1/16}{16} = 1 - \frac{1}{256}$

$= \frac{255}{256}$

(e) Write the definite integral for the moment generating function of X.

Be sure your integral is a proper integral.

$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^2 \frac{1}{4}x^3 e^{tx} dx$.

Discrete Distributions

Distribution	$f(x)$	S	$M(t)$	μ	σ^2
Bernoulli	$p^x(1-p)^{1-x}$	$\{0,1\}$	$(1-p)+pe^t$	p	$p(1-p)$
Binomial $b(n,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$	$\{0,1,\ldots,n\}$	$\left((1-p)+pe^t\right)^n$	np	$np(1-p)$
Geometric	$(1-p)^{x-1}p$	$\{1,2,\ldots\}$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
Hypergeometric	$\dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$	$n-N_2 \leq x \leq N_1, x \leq n$		$n\dfrac{N_1}{N}$	$n\dfrac{N_1}{N}\dfrac{N_2}{N}\dfrac{N-n}{N-1}$
Negative Binomial	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$	$\{r,r+1,r+2,\ldots\}$	$\left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$	$\dfrac{r}{p}$	$\dfrac{r(1-p)}{p^2}$
Poisson	$\dfrac{\lambda^x e^{-\lambda}}{x!}$	$\{0,1,2,\ldots\}$	$e^{\lambda(e^t-1)}$	λ	λ
Uniform	$\dfrac{1}{m}$	$\{1,2,\ldots,m\}$		$\dfrac{m+1}{2}$	$\dfrac{m^2-1}{12}$

Continuous Distributions

Distribution	$f(x)$	S	$M(t)$	μ	σ^2
Exponential	$\dfrac{1}{\theta}e^{-x/\theta}$	$x \geq 0$	$\dfrac{1}{1-\theta t}$	θ	θ^2
Uniform $U(a,b)$	$\dfrac{1}{b-a}$	$a \leq x \leq b$	$\dfrac{e^{bt}-e^{at}}{(b-a)t}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$