Chapter 5 Solutions

1) a) Object: real number x
Property:
Something: $f(x) \leq f(x^*)$

b) Object: element x
Property: $x \in S$
Something: $g(x) \geq f(x)$

c) Object: element x
Property: $x \in S$
Something: $x \leq u$

2) a) Object: real numbers x and y
Property: $x<y$
Something: $f(x) < f(y)$

b) Object : elements x,y and real number t
Property: $x,y \in C$ and $0 \leq t \leq 1$
Something: $tx + (1 - t)y \in C$

c) Object: real numbers x and y, real number t
Property: $0 \leq t \leq 1$
Something: $f(tx + (1 - t)y) \leq t\, f(x) + (1 - t)\, f(y)$

4) a) If p is a prime number, then p+7 is a composite.
b) If A, B, and C are sets with $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
c) If p and q are integers with $q \neq 0$, then p/q is rational.

5) a) Choose a real number x\`
Show that $f(x`) \leq f(x^*)$
b) Choose an element x\`
Show that $g(x`) \geq f(x`)$
c) Choose an element $x` \in S$
Show that $x` \leq u$.

6) a) Choose real numbers x\` and y\` , with $x` < y`$
Show that $f(x`) < f(y`)$

b) Choose elements $a,b \in C$, and t\` with $0 \leq t` \leq 1$
Show that $t`a + (1 - t`)b \in C$

c) Choose real numbers x\` and y\`, and t\` with $0 \leq t` \leq 1$.
Show that: $f(t`x` + (1-t`)y`) \leq t`f(x`) + (1-t`)f(y`)$.

8) Key Question: How can I show that a set is a subset of another set.
Answer: Show that every element of the subset is an element of the set.

Need to show that:

B_1: For all $r \in R$, $r \in T$.

The forward process would begin with:

A_1: Choose an element $r` \in R$.

Must show that,

B_2: $r` \in T$.

10) Key Question: How do you show that a function is increasing?
Answer: $f(y) > f(x)$ for all x,y with $x < y$.

A_1: Choose real numbers x' and y', with $x' < y'$.

To show,

B_2: $f(x') < f(y')$.

13) a) Incorrect: x` and y` should be chosen $\in$ S, not T.
b) Correct
c) Incorrect: A_1 should be $(x`,y`) \in S$
B_1 should be $(x`, y`) \in T$
d) Incorrect: the values should be general, not specific
e) Correct, but it would be better to use different letters than x and y.

15) The choose method is used in the first sentence of the proof: "Let x be a real number".
The key question "How can I show that a real number is a maximizer of a function?"

In order to answer this it needs to be shown that:

B1: For all real numbers x, $f(x^*) \geq f(x)$.

There are two cases that can be addressed here.
Case 1:

A_1: $x^* \geq x$
A_2: $x^* - x \geq 0$
A_3: $a(x^* + x) + b \geq 0$. Check that this is true!
(Algebraically using $x^* = -b/2a$ from the proposition.)

Then, using in the function $f(x) = ax^2+bx+c$, we get: $a(x^{*\wedge}2 – x^2) + bx^* - bx + c > c$
Arranging the terms on their proper sides yields:

$ax*^2 + bx* + c > ax^2 + bx + c$

This is B_1: f(x*) > f(x), therefore the proof is complete.

Case 2 can be done similarly for when x* < x.

16) Choose method: The choose method is used in the second sentence: "let $x \in R \cap S$." The choose method is used here because in order to show that $R \cap S \subseteq T$, it is required to show that for all $x \in R \cap S$, $x \in T$.

18)

A_1: $t \in T$	From the choose method
A_2: $x \in S$	From the choose method.
A_3: $x(x - 3) \leq 0$	
A_4: $x \geq 0$, $x - 3 \leq 0$.	A_3 and A_4 follow from the definition of S (note $x \leq 0$, $x-3 \geq 0$ cannot happen – why?)
A_5: $0 \leq x \leq 3$	This is gotten working forward from A_4.
A_6: $t \geq 3$	This is given as a condition of set T in the Proposition.

A_7: Combining A_5 and A_6 yields $x \leq 3 \leq t$, showing $x \leq t$.

A_8: Showing that $x \leq t$ completes the proof as all elements of S are now lower than all elements of T, showing that all elements $t \in T$ are upper bounds for the set S.

19) **Analysis of Proof:**

The appearance of the quantifier "for all" in the conclusion suggests the choose method.

The choose method is used in the first sentence of the proof in saying that r,q are the same sign and p/(qr)>0, and that p is a positive integer.

A_1: Let q and r have the same sign, $q,r \neq 0$. Let p be a positive integer.

A_2: Then $p/(qr)>0$.	This is true because p is a positive integer and q,r have the same sign therefore the denominator will be positive.
A_3: $q<r$	This is given in the hypothesis.

A_4: Multiplying both sides of the inequality in A_3 by p/(qr) gives:

$$(pq/qr) < (pr/qr)$$

A_5: This reduces to $(p/r) < (p/q)$.

Therefore the proof is complete.

22) **Key Question**: How can I show that a set is a convex set?

Answer (B_1): For every x,y ∈ C, and for every real number t with $0 \leq t \leq 1$, $t(x) + (1 - t)(y) \in C$. (From the definition of a convex set.)

A_1: Let x',y' ∈ C, t' with $0 \leq t' \leq 1$.

For which it must be shown that,

B_2: $t'(x') + (1 - t')(y') \in C$, meaning that $a(tx + (1 - t)y) \leq b$. (Because set C is real numbers x: $ax \leq b$)

Because x and y are in C, it can be shown that

A_2: $ax' \leq b$, and $ay' \leq b$.

Multiplying A_2 by t' and (1-t') and adding the two inequalities gives:

A_3: $t'ax' + (1 - t')ay' \leq t'b' + (1 - t')b' = b'$

Proof: Let x',y' ∈ C, and t' be a real number with $0 \leq t' \leq 1$. From the definition of C: $ax' \leq b$, and $ay' \leq b$. Multiplying each of these inequalities by both $t' \geq 0$ and $(1-t') \geq 0$, respectively, and performing algebra yields $a[t'x' + (1 - t')y'] \leq b$. By showing this, it means that $t(x') + (1 - t)(y') \in C$, from the definition of a convex set. Therefore, the set C is a convex set and the proof is complete.