MATH 300 (Section 001)
Prof. Meade

Exam 3
15 April 2014

University of South Carolina
Spring 2014

Name: Key

Instructions:

1. There are a total of 7 problems on 4 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. To be eligible for full credit, explanations and justifications must be written in complete English sentences.

Problem	Points	Score
1	10	
2	20	
3	10	
4	20	
5	10	
6	10	
7	20	
Total	100	

Good Luck!

1. (10 points)

   Definition. A set of real numbers S is **bounded** if and only if there is a real number $M > 0$ such that, $\forall$ elements $x \in S$, $|x| < M$.

   Identify the object, the certain property, and the something that happens for each quantifier as they appear left to right in the above definition.

   $\exists$: O: real number M
   CP: $M>0$
   SH: $\forall x \in S, |x|<M$

   $\forall$: O: x
   CP: $x \in S$
   SH: $|x|<M$

2. (20 points)

   Proposition: For all real numbers x and y with $x < y$, there is a rational number r such that $x < r < y$.

   Proof. Let x and y be real numbers with $x < y$. Then let $\epsilon = y - x > 0$ and so, from a proposition in Exercise 7.11 (which you proved in the homework), there is an integer $n > 0$ such that $n\epsilon > 2$. Now let m be an integer with $nx < m < ny$. Then the desired rational number is $r = \frac{m}{n}$ and so the proof is complete.

   Answer the following questions about the preceding proof.

   (a) What proof technique is used in the first sentence? Why is this technique chosen? Explain. Choose, because the first quantifier is a $\forall$

   (b) What is being done in the second sentence?
   Working forward from $y>x$ to begin construction of r
   (also: specialization of Exercise 7.11 to $a=2$)

   (c) What proof technique is being used in the last sentence of the proof? Why is this technique being used? Explain.
   Construction, because inner quantifier is a $\exists$

   (d) Is the claim that the proof is complete in the last sentence justified? Why or why not?
   Yes. The proof does construct an r that works for any choice of x & y.

3. (10 points)

Definition. A sequence x_1, x_2, ... of real numbers is **increasing** if and only if for every integer $k = 1, 2, \ldots, x_k < x_{k+1}$.

Write the negation (NOT) of the following definition in such a way that the word "not" does not appear in the statement after the words "if and only if".

a sequence $x_1, x_2, \ldots$ of real numbers is not increasing if and only if there is an integer $k = 1, 2, \ldots$ such that $x_k \geq x_{k+1}$.

4. (20 points)

Proposition: Suppose that m and n are integers. If either mn is divisible by 4 or n is not divisible by 4, then n is an even integer and m is an odd integer.

(a) What statement(s) will you work forward from if you want to prove "NOT B implies NOT A"?

n is an odd integer or m is an even integer

(b) What statement(s) will you work backward from if you want to prove "NOT B implies NOT A"?

mn is not divisible by 4 and n is divisible by 4

5. (10 points)

Proposition: If f and g are two functions such that (1) $g \geq f$ and (2) f is unbounded above, then g is unbounded above.

When applying the contradiction method to prove this proposition, what should you assume?

f and g are functions with 1) $g \geq f$,
2) f is unbounded above,
and
3) g is bounded above.

6. (10 points)

   Proposition: If m does not divide n, then $mx^2 + nx + (n - m)$ has no positive integer root.

   In a proof by the contrapositive method, which of the following is a result of the forward process? Explain your answer.

   (a) m divides n
   (b) There is an integer $x > 0$ such that $mx^2 + nx + (n - m) \neq 0$.
   (c) There is an integer $x > 0$ such that $mx^2 + nx + (n - m) = 0$.
   (d) There is an integer $x \leq 0$ such that $mx^2 + nx + (n - m) = 0$.

A contrapositive proof begins with the assumption that:
$mx^2+nx+(n-m)$ has a positive integer root
Statement (c) says exactly this.

7. (20 points)

   Proposition: If n and p are positive integers and $n|p$, then $n \leq p$.
   Proof. Suppose $n > p$. Then because $n|p$, there is an integer c such that $p = cn$. Now $n > 0$ and $p > 0$, so $c > 0$. Therefore, $p = cn > cp \geq p$, and so the proof is complete.

   (a) Is the contradiction or contrapositive proof technique being used?

Contradiction (because the proof starts with the assumption that $n|p$ and $n>p$.)

   (b) If the contradiction method is used, identify the contradiction.

The contradiction is that no number is strictly less than itself ($p<p$ is false!)