MATH 300 (Section 001)
Prof. Meade

University of South Carolina
Spring 2014

Exam 1
18 February 2014

Name: Key

Instructions:

1. There are a total of 9 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. To be eligible for full credit, explanations and justifications must be written in complete English sentences.

Problem	Points	Score
1	20	
2	8	
3	12	
4	8	
5	12	
6	12	
7	12	
8	8	
9	8	
Total	100	

Good Luck!

1. (20 points) Prepare a truth table for each of the following statements.

(a) $\sim (A \vee B)$

A	B	$A \vee B$	$\sim(A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(b) $(\sim A) \wedge (\sim B)$

A	B	$\sim A$	$\sim B$	$(\sim A) \wedge (\sim B)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(c) $((\sim B) \wedge (A \Rightarrow B)) \Rightarrow \sim A$

A	B	$\sim B$	$A \Rightarrow B$	$(\sim B) \wedge (A \Rightarrow B)$	$\sim A$	$((\sim B) \wedge (A \Rightarrow B)) \Rightarrow \sim A$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

(d) In general, how can you tell if two statements are logically equivalent?

Two statements are logically equivalent when they have the same truth values.

(e) Which, if any, of the above statements are logically equivalent?

The statements in (a) & (b) have the same truth values.

$\sim (A \vee B)$ is logically equivalent to $(\sim A) \wedge (\sim B)$.

2. (8 points) Prove that the following statement is false: *If n is prime, then n^2+n+7 is prime.* HINT: *Find a counterexample.*

The first few primes are: 2, 3, 5, 7, 11 (yes, 2 is a prime number!)

n	n^2+n+7
2	13 — prime
3	19 — prime
5	37 — prime
7	63 — composite!
11	

Since $7^2+7+7=63$ is composite, & 7 is prime, n^2+n+7 is not prime whenever n is prime.

3. (12 points) Write the indicated statement for each of the following propositions.

$A \Rightarrow B$
inverse: $B \Rightarrow A$
converse: $\sim A \Rightarrow \sim B$
contrapos: $\sim B \Rightarrow \sim A$

   (a) The converse of "If n is an integer for which n^2 is odd, then n is odd."

   If n is an odd integer, then n^2 is odd.

   (b) The contrapositive of "If r is a real number such that $r^2 = 2$, then r is not rational."

   If r is a real number that is rational, then $r^2 \neq 2$.

   or If r is a rational number, then $r^2 \neq 2$.

   (c) The inverse of "If $n > 1$ is an integer for which $2^n - 1$ is prime, then n is prime".

   If $n > 1$ is an integer for which $2^n - 1$ is composite, not prime
   then n is composite
   not prime.

4. (8 points) Suppose you know that

   Proposition. *If the right triangle RST with sides of lengths r and s and hypotenuse of length t satisfies* $t = \sqrt{2rs}$, *then the triangle RST is isosceles.*

   What would you have to show in order to use the above proposition to prove "If the right triangle ABC with side of lengths a and b and hypotenuse of length c has an area of $c^2/4$, then the triangle is isosceles."

   To apply the Proposition to the Δ ABC we need to show that $c = \sqrt{2ab}$ (match: $r \to a$, $s \to b$ or vice versa, $t \to c$)

5. (12 points) For each of the following statements, obtain a new statement in the backward process by using a definition to answer the key question. If necessary, rewrite the definitions so that there is no overlapping notation.

   (a) The integer $m > 1$ is prime.

   m is an integer greater than 1 that is divisible only by 1 and by m.

   (b) triangle ABC is equilateral

   the ΔABC has three sides of equal length

   (c) $\sqrt{n}$ is rational (n is an integer)

   $\sqrt{n} = \frac{p}{q}$ where p and q are integers and $q \neq 0$.

6. (12 points) Use a definition to work forward one step from the hypothesis.

   (a) If n is an integer greater than 1 for which $2^n - 1$ is prime, then n is prime.

   Because 2^n-1 is prime, 2^n-1 is divisible only by 1 and by 2^n-1.

   (b) If a, b, and c are integers for which $a \mid b$ and $b \mid c$, then $a \mid c$.

   Because $a \mid b$ and $b \mid c$ there are integers p and q for which $b = pa$ and $c = qb$.

   (c) If the quadrilateral $ABCD$ is a parallelogram with one right angle, then $ABCD$ is a rectangle.

   Because quadrilateral ABCD is a parallelogram, both pairs of opposite angles are congruent.

   *There are other answers that could be acceptable for some of these, particularly (c).

7. (12 points) Consider the following statement:

If n is an odd integer, then $n^2 + 1$ is an even integer.

(a) Pose a key question.

How can we show that an integer is even?

(b) Then use a definition to answer the question abstractly.

To show an integer is even, we show it can be written as twice some integer.

(c) And, finally, use that same definition to apply the answer to the specific problem.

We need to show that $n^2+1 = 2k$ for some integer k.

8. (8 points) Prove: *Let a and b be integers. If $a \mid b$, then $a^2 \mid b^2$.*

1 H1: $b = ka$ for some integer k.

4 A1. $b^2 = (ka)^2 = k^2 a^2$

so, $l = k^2$.

3 B1: $b^2 = l a^2$ for some integer l.

2 $\therefore$ $a^2 \mid b^2$

Proof: Let a and b be integers.
Because $a \mid b$ we know there is an integer k such that $b = ka$.
Then $b^2 = (ka)^2 = k^2 a^2 = l a^2$ where $l = k^2$
Since is also integer, this means $a^2 \mid b^2$. $\boxed{\Sigma}$

9. (8 points) Provide an analysis of the following proof.

> *If n is an integer greater than 2, a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse, then $c^n > a^n + b^n$.*
>
> **Proof.** You have that $c^n = c^2c^{n-2} = (a^2+b^2)c^{n-2}$. Observing that $c^{n-2} > a^{n-2}$ and $c^{n-2} > b^{n-2}$, it follows that $c^n > a^2(a^{n-2}) + b^2(b^{n-2})$. Consequently, $c^n > a^n + b^n$. ■

Let a & b be the lengths of the legs of a right triangle,
Let c be the length of the hypotenuse of this triangle.
Let $n > 2$.

Following the derivation:

$c^n = c^2 c^{n-2}$ (properties of exponents)

$= (a^2+b^2)c^{n-2}$ (Pythagorean Thm: $c^2 = a^2+b^2$)

$= a^2c^{n-2} + b^2c^{n-2}$ (algebra)

$> a^2a^{n-2} + b^2b^{n-2}$ (because $c > a$ & $n-2 > 0 \Rightarrow c^{n-2} > a^{n-2}$ & $b > a$ & $n-2 > 0 \Rightarrow c^{n-2} > b^{n-2}$)

$= a^n + b^n$ (algebra of exponents)