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sec1-1.mws

Modeling Via Differential Equations

Section 1.1 - Blanchard, Devaney, Hall

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Introduction

The purpose of this worksheet is to introduce you to some of the ways Maple can be used to assist with the analysis of differential equations. You are not expected to currently understand everything that is presented in this worksheet. As the course progresses you will be expected to become proficient at these types of "computations".

This worksheet is intended to be use while reading section 1.1 of the text. The definitions are not repeated in this worksheet. There are a few places where this worksheet differs from the text. These are clearly identified. There are a number of questions for you to answer as you work through this worksheet. Please take the time to do what is asked. Ask questions as they arise.

Getting Started

Every Maple worksheet should begin by re-initializing the Maple "kernel" and loading the additional packages that we are most likely to use.

```
> restart;
> with( plots ):
> with( DEtools ):
>
```

Unlimited Population Growth

One model for unlimited population growth is represented by the following first-order ordinary differential equation.

```
> MODEL := diff( P(t), t ) = k * P(t);
```

Question

Equilibrium Solution

(See the text for a definition of equilibrium solution.)

```
> EQUILIBRIUMeqn := rhs( MODEL ) = 0;
> EQUILIBRIUMsol := solve( EQUILIBRIUMeqn, { P(t) } );
```

Qualitative Analysis

```
> IC := P(0) = p[0];
```

The text provides a method by which we can determine that solutions to this model are increasing if $k > 0$ and $p_0 > 0$ and decreasing if $k < 0$ and $p_0 > 0$. This type of analysis can be done for other equations but is often quite difficult.

```
>
```

Analytic Solution

We will learn specific techniques for solving special classes of differential equations. The Maple command for finding explicit solutions to differential equations is dsolve. The arguments are the initial value problem (a set containing the differential equation and the initial condition) and the function for which we are trying to solve.

```
> SOLN := dsolve( { MODEL, IC }, P(t) );
```

Warning

```
>
```

The U.S. Population

The computations used to determine values for the constants k and p_0 from the census data are easily reproduced in Maple. The idea is that two data points can be used to construct a system of two equations for the two unknown constants. While it is possible to exactly duplicate the steps shown in the text to determine k and p_0, it is simpler to ask Maple to solve the system.

```
>
> EQ1790 := subs( P(t)=3.9, t=0, SOLN );
> EQ1800 := subs( P(t)=5.3, t=10, SOLN );
> CONST := solve( { EQ1790, EQ1800 }, { k, p[0] } );
```

Thus, this model predicts the U.S. population is given by the following function.

```
> solP := subs( CONST, rhs(SOLN) );
>
```

To evaluate the reasonableness of this prediction we will often plot the function. However, since the text provides a table with census values from 1790 through 2050, let's produce predicted populations for each decade (and a few decades into the next millenium).

```
> t := 't':
> for t from 0 to 260 by 10 do
>   printf( `%4d  %10.4f\n`, t+1790, solP );
> od;
> t := 't':
>
```

Question

Question

```
>
```

Logistic Population Growth

The logistic population model is motivated and derived in the text (pp. 8-9). The resulting differential equation is

```
> MODELlog := diff( P(t), t ) = k * ( 1 - P(t)/N ) * P(t);
```

with the initial condition

```
> IClog := P(0) = p[0];
>
```

Equilibrium Solution

The equilibrium solutions are found in the same manner as in the first example: determine all constants for which the RHS of the model is zero.

```
> EQUILIBRIUMeqn := rhs( MODELlog ) = 0;
> EQUILIBRIUMsol := solve( EQUILIBRIUMeqn, { P(t) } );
```

Note that there are now two equilibria. In addition to the zero solution, the *carrying capacity* of the population, *N*, is an equilibrium solution.

```
>
```

Qualitative Analysis

The text gives a good description of the qualitative analysis of the logistic model. Assume $k > 0$. The basic idea is that (for equations of this type - we'll be more specific shortly) the solutions are either always increasing or always decreasing in regions between the equilibria. In this case, for $0 < P < N$, the RHS of the model is positive $k\left(1 - \frac{P}{N}\right)P > 0$, and so the solutions are increasing (to the carrying capacity). By the same reasoning, for $P > N$ or $P < 0$, $k\left(1 - \frac{P}{N}\right)P < 0$ and solutions are decreasing (to the carrying capacity).

Questions

```
>
```

Analytic Solution

We will learn specific techniques for solving special classes of differential equations. The Maple command for finding explicit solutions to differential equations is dsolve. The arguments are the initial value problem (a set containing the differential equation and the initial condition) and the function for which we are trying to solve.

```
> SOLNlog := dsolve( { MODELlog, IC }, P(t) );
>
```

The U.S. Population

To conclude this investigation of the models, lets determine the values of the parameters that fit the U.S. population in 1790, 1900, and 1990. (Why are three years used for this fit?)

```
> EQ1790 := subs( P(t)=3.9, t=0, SOLNlog );
> EQ1900 := subs( P(t)=75, t=110, SOLNlog );
> EQ1990 := subs( P(t)=249, t=200, SOLNlog );
>
```

The solution(s) to this system of three equations and three unknowns can be found using the solve command.

```
> solve( { EQ1790, EQ1900, EQ1990 }, { k, N, p[0] } );
```

Note that while there are many solutions, all but one of them involves complex numbers. Since we know our answer should be real-valued (WHY?) we can assume that these are the parameter values that we should use for this problem. There is a fancy way to have Maple select all real-valued solutions but it's simpler to simply copy and paste the portion of the output that interests us.

```
> CONST := {p[0] = 3.900000000, k = .2934545722e-1, N =
  302.9459184};
```

(optional) Fancy method of extracting real-valued solution from multiple solutions of a system of equations

```
>
```

The logistic model for the U.S. population based on the 1790, 1900, and 1990 populations is

```
> solPlog := subs( CONST, rhs( SOLNlog ) );
>
```

A table of predicted populations for 1790 - 2050 is easily created as before

```
> t := 't':
> for t from 0 to 260 by 10 do
>   printf( `%4d  %10.4f  %10.4f\n`, t+1790, solP, solPlog );
> od;
> t := 't':
>
```

Note that the logistic model does a much better job of following the actual U.S. populations (as reported in Table 1.1 on p. 7).

A final comparison of the two solutions can be performed by looking at the plot of the solutions to each model.

```
> plot( [ solP, solPlog ], t=0..300, 0..500, title=`Unlimited and
  Logistic Growth of the US Population (in millions)` );
```

Note how one solution appears to level off around 300 (million) and the other appears to continue to grow forever. Which curve corresponds with which model?

```
>
```