```
>
```

BDH3.mws

This worksheet contains the direction fields and analysis for the in-class examples presented during the discussion of the material in Chapter 3 of the text.

```
>
> restart;
> with(DEtools):
> with(plots):
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
>
```

Section 3.1 -- Properties of Linear Systems and The Linearity Principle

Recreation of Figures 3.1 - 3.4 (pp. 226 - 227)

Figures 3.1 and 3.3

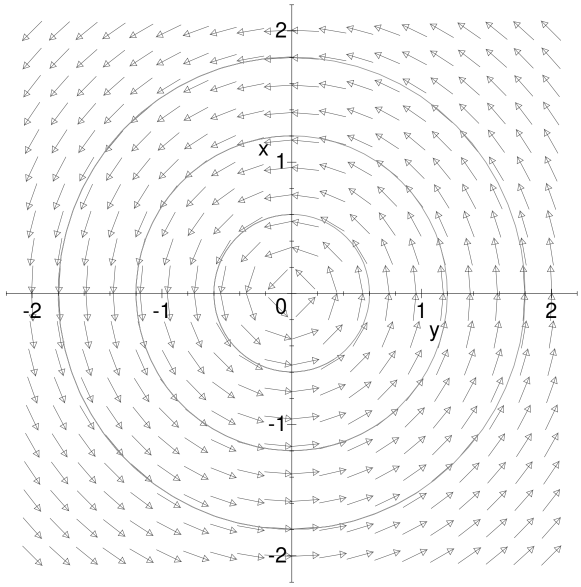
```
> ode1 := diff( x(t), t ) = y(t);
> ode2 := diff( y(t), t ) = -x(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t)$$

$$ode2 := \frac{\partial}{\partial t}\mathrm{y}(t) = -\mathrm{x}(t)$$

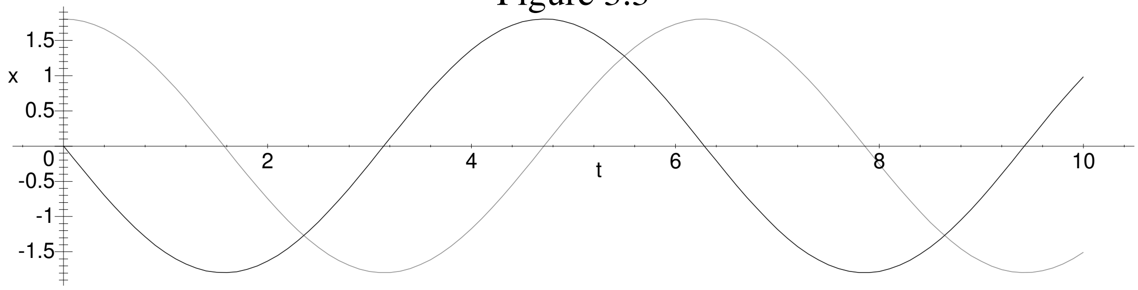
```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..10:
> RANGE := x=-2..2, y=-2..2:
> IC := [ [ x(0)=0.6, y(0)=0 ], [ x(0)=1.2, y(0)=0 ], [ x(0)=1.8, y(0)=0
  ] ]:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> #dirPLOT;
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
> display( [ dirPLOT, solPLOT ], title=`Figure 3.1` );
```

Figure 3.1




```
> 
> xPLOT := DEplot( MODEL, VARS, DOMAIN, [ IC[3] ], scene=[t,x],
  arrows=NONE, stepsize=0.1, linecolor=GREEN ):
> yPLOT := DEplot( MODEL, VARS, DOMAIN, [ IC[3] ], scene=[t,y],
  arrows=NONE, stepsize=0.1, linecolor=BLUE ):
> display( [ xPLOT, yPLOT ], title=`Figure 3.3` );
```

Figure 3.3




```
> 
```

Figures 3.2 and 3.4

```
> ode1 := diff( x(t), t ) = -x(t) + 4*y(t);
> ode2 := diff( y(t), t ) = -3*x(t) - y(t);
> MODEL := { ode1, ode2 }:
```

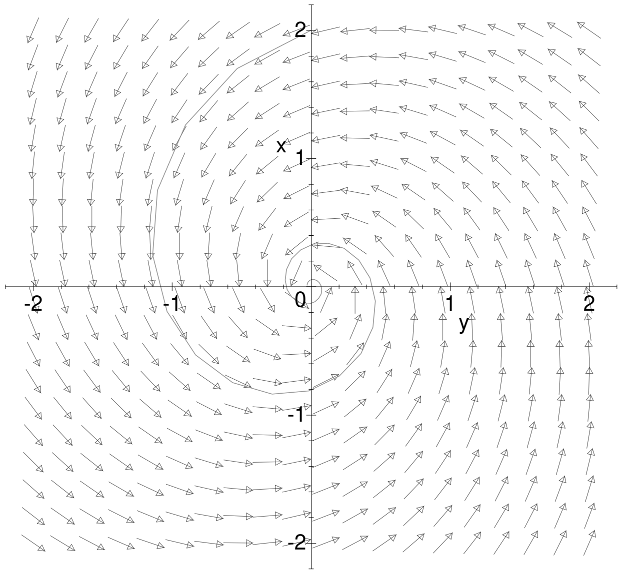
$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = -\mathrm{x}(t) + 4\,\mathrm{y}(t)$$

$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = -3\,\mathrm{x}(t) - \mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..4:
> RANGE := x=-2..2, y=-2..2:
> IC := [ [ x(0)=2, y(0)=0 ] ]:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> #dirPLOT;
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
```

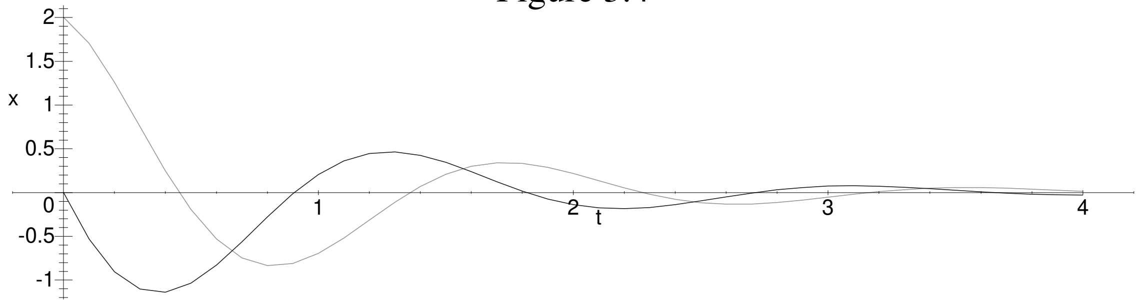
```
> display( [ dirPLOT, solPLOT ], title=`Figure 3.2` );
```

Figure 3.2




```
>
> xPLOT := DEplot( MODEL, VARS, DOMAIN, IC, scene=[t,x], arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> yPLOT := DEplot( MODEL, VARS, DOMAIN, IC, scene=[t,y], arrows=NONE,
  stepsize=0.1, linecolor=BLUE ):
> display( [ xPLOT, yPLOT ], title=`Figure 3.4` );
```

Figure 3.4




```
>
```

Section 3.2 -- Straight-Line Solutions

Example with $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

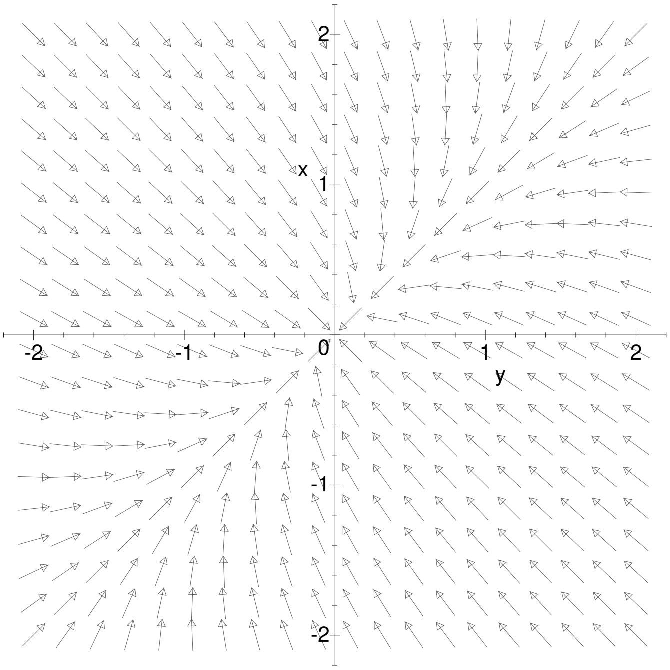
```
> ode1 := diff( x(t), t ) = -2*x(t) + y(t);
> ode2 := diff( y(t), t ) = x(t) - 2*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = -2\,\mathrm{x}(t) + \mathrm{y}(t)$$

$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = \mathrm{x}(t) - 2\,\mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..1:
> RANGE := x=-2..2, y=-2..2:
```

```
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
```

Can you SEE both of the straight-line solutions in this direction field? Let's see if this is consistent with the eigenvalues and eigenvectors analysis:

```
> A := matrix( [ [ -2, 1 ], [ 1, -2 ] ] );
```

$$A := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 + 4\,\lambda + 3$$

```
> eigenvals( A );
```

$$-3, -1$$

```
> eigenvects( A );
```

$$[-1, 1, \{[1, 1]\}], [-3, 1, \{[-1, 1]\}]$$

```
>
```

Thus, the straight-line solutions are:

```
> Y1 := exp(-3*t) * matrix( 2, 1, [ -1, 1 ] );
```

$$Y1 := \mathbf{e}^{(-3\,t)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

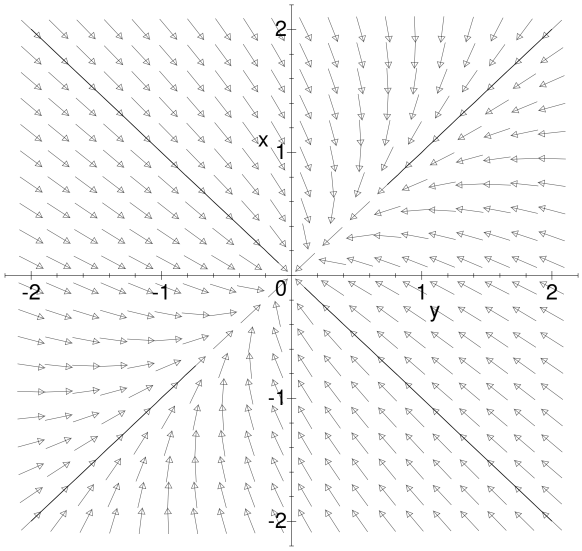
```
> Y2 := exp(-t) * matrix( 2, 1, [1,1] );
```

$$Y2 := \mathbf{e}^{(-t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>
> ICline := [ [ x(0)=2, y(0)=2 ], [ x(0)=2, y(0) = -2 ], [ x(0)=-2, y(0)=2
  ], [ x(0)=-2, y(0)=-2 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  stepsize=0.1, linecolor=BLUE ):
> #linePLOT;
> display( [ dirPLOT, linePLOT ], title=`Direction Field & Straight-Line
```

```
  Solns` );
```

Direction Field & Straight-Line Solns



The fact that both approach the origin is consistent with the fact that the eigenvalues are negative. In addition, notice how the solutions in the second- and fourth-quadrants approach the origin more rapidly than those in the first- and third-quadrants. What explains this phenomenon?

```
>
```

Section 3.3 -- Phase Planes for Linear Systems with Real Eigenvalues

Saddle

```
> A := matrix( [ [ 3, 0 ], [ 1, -2 ] ] );
```

$$A := \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$(\lambda - 3)(\lambda + 2)$$

```
> eigenvals( A );
```

$$3, -2$$

Since there is one positive and one negative eigenvalue, the origin is a saddle for this system.

```
>
> eigenvects( A );
```

$$[-2, 1, \{[0, 1]\}], [3, 1, \{[5, 1]\}]$$

```
>
```

The straight-line solutions are:

```
> Y1 := exp(-2*t) * matrix( 2, 1, [ 0, 1 ] );
```

$$Y1 := \mathbf{e}^{(-2\,t)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
> Y2 := exp(3*t)  * matrix( 2, 1, [ 5, 1 ] );
```

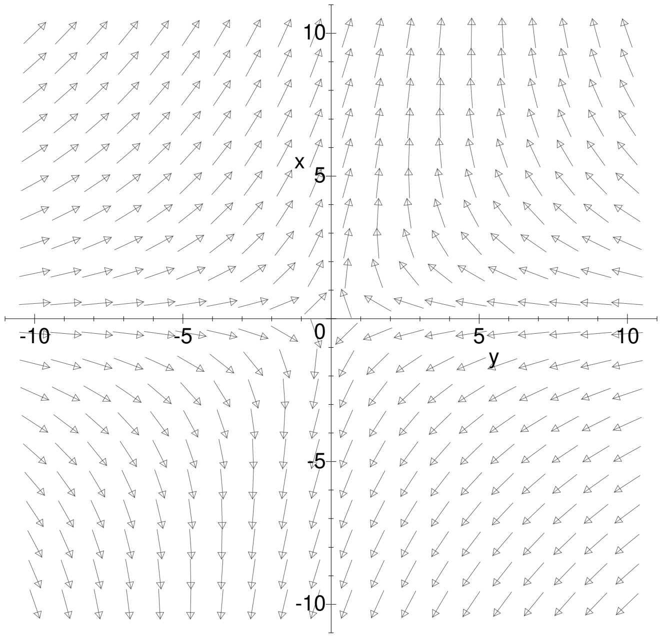
$$Y2 := \mathbf{e}^{(3\,t)} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

```
>
> ode1 := diff( x(t), t ) = 3*x(t);
> ode2 := diff( y(t), t ) =   x(t) - 2*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = 3\,\mathrm{x}(t)$$

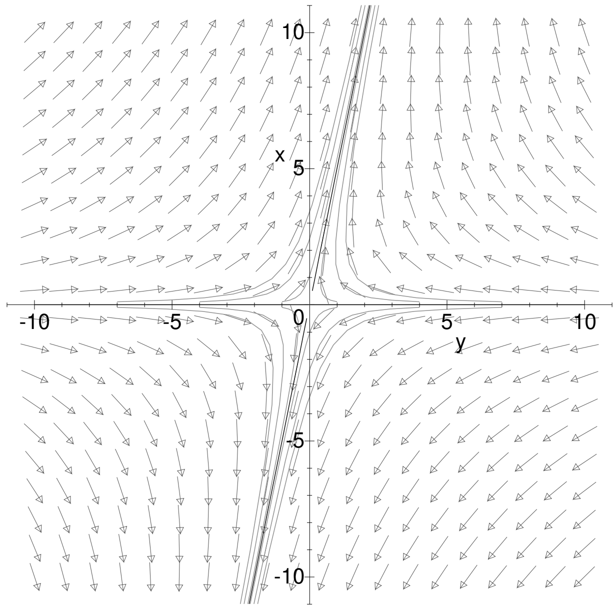
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = \mathrm{x}(t) - 2\,\mathrm{y}(t)$$

```
>
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..3:
> RANGE := x=-10..10, y=-10..10:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=0.1, y(0)=1 ], [ x(0)=0.1, y(0)=4 ], [ x(0)=0.1, y(0)=7 ],
          [ x(0)=-0.1, y(0)=1 ], [ x(0)=-0.1, y(0)=4 ], [ x(0)=-0.1, y(0)=7
  ],
          [ x(0)=0.1, y(0)=-1 ], [ x(0)=0.1, y(0)=-4 ], [ x(0)=0.1, y(0)=-7
  ],
          [ x(0)=-0.1, y(0)=-1 ], [ x(0)=-0.1, y(0)=-4 ], [ x(0)=-0.1,
  y(0)=-7 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ x(0)=0, y(0)=10 ], [ x(0)=0, y(0)=-10 ],
              [ x(0)=0.5, y(0)=0.1 ], [ x(0)=-0.5, y(0)=-0.1 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Example of Saddle` );
```

Example of Saddle



Notice how these solutions move (essentially) straight down the y-axis before moving in the direction of the eigenvector corresponding to the positive eigenvalue.

```
>
```

Source

```
> A := matrix( [ [ 2, 2 ], [ 1, 3 ] ] );
```

$$A := \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 - 5\,\lambda + 4$$

```
> eigenvals( A );
```

$$1, 4$$

With two positive eigenvalues, the origin is a source for this system.

```
>
> eigenvects( A );
```

$$[1, 1, \{[-2, 1]\}], [4, 1, \{[1, 1]\}]$$

```
>
```

The straight-line solutions are:

```
> Y1 := exp(4*t) * matrix( 2, 1, [ 1, 1 ] );
```

$$Y1 := \mathbf{e}^{(4\,t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
> Y2 := exp(t)   * matrix( 2, 1, [ -2, 1 ] );
```

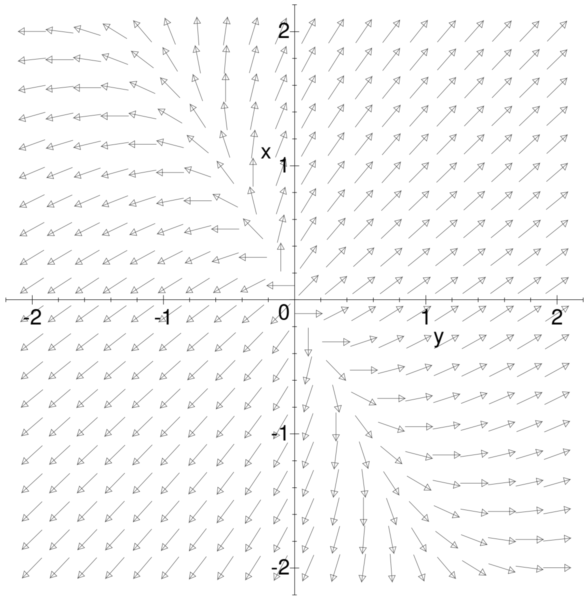
$$Y2 := \mathbf{e}^{t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

```
>
> ode1 := diff( x(t), t ) = 2*x(t) + 2*y(t);
> ode2 := diff( y(t), t ) =   x(t) + 3*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = 2\,\mathrm{x}(t) + 2\,\mathrm{y}(t)$$

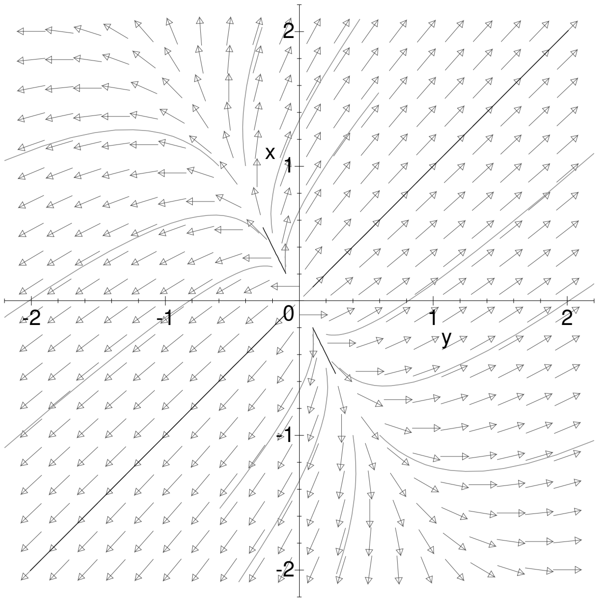
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = \mathrm{x}(t) + 3\,\mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..1:
> RANGE := x=-2..2, y=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=0.25, y(0)=-0.1 ], [ x(0)=0.5, y(0)=-0.2 ], [ x(0)=1,
  y(0)=-0.4 ],
          [ x(0)=0.25, y(0)=-0.2 ], [ x(0)=0.5, y(0)=-0.3 ], [ x(0)=1,
  y(0)=-0.6 ],
          [ x(0)=-0.25, y(0)=0.1 ], [ x(0)=-0.5, y(0)=0.2 ], [ x(0)=-1,
  y(0)=0.4 ],
          [ x(0)=-0.25, y(0)=0.2 ], [ x(0)=-0.5, y(0)=0.3 ], [ x(0)=-1,
  y(0)=0.6 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ x(0)=0.1, y(0)=0.1], [ x(0)=-0.1, y(0)=-0.1 ],
              [ x(0)=0.2, y(0)=-0.1 ], [ x(0)=-0.2, y(0)=0.1 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Example of Source` );
```

Example of Source



Notice how the solutions become parallel to the eigenvector for the larger eigenvalue. The different sizes of the eigenvalues can also be seen by noting the relative length of the two straight-line solutions.

```
>
```

Sink

```
> A := matrix( [ [ -2, -2 ], [ -1, -3 ] ] );
```

$$A := \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 + 5\,\lambda + 4$$

```
> eigenvals( A );
```

$$-4, -1$$

With two negative eigenvalues, the origin is a sink for this system.

```
>
> eigenvects( A );
```

$$[-1, 1, \{[-2, 1]\}], [-4, 1, \{[1, 1]\}]$$

```
>
```

The straight-line solutions are:

```
> Y1 := exp(-t) * matrix( 2, 1, [ -2, 1 ] );
```

$$Y1 := \mathbf{e}^{(-t)} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

```
> Y2 := exp(-4*t) * matrix( 2, 1, [ 1, 1 ] );
```

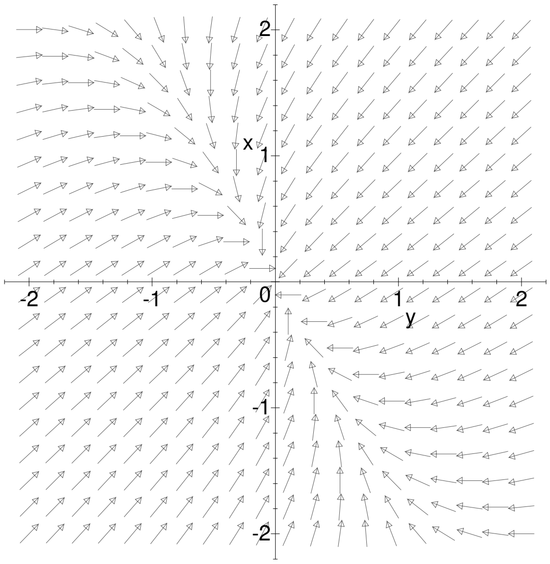
$$Y2 := \mathbf{e}^{(-4\,t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>
> ode1 := diff( x(t), t ) = -2*x(t) - 2*y(t);
> ode2 := diff( y(t), t ) =   -x(t) - 3*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = -2\,\mathrm{x}(t) - 2\,\mathrm{y}(t)$$

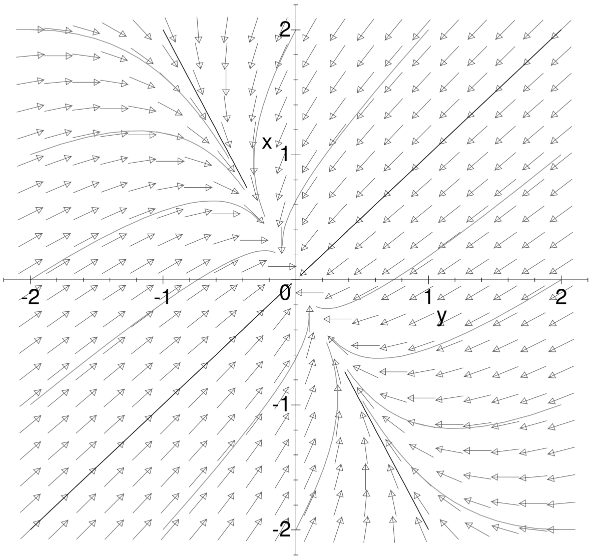
$$ode2 := \frac{\partial}{\partial t}\, y(t) = -x(t) - 3\, y(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..1:
> RANGE := x=-2..2, y=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=1, y(0)=2 ], [ x(0)=0, y(0)=2 ], [ x(0)=-1, y(0)=2 ],
          [ x(0)=1, y(0)=-2 ], [ x(0)=0, y(0)=-2 ], [ x(0)=-1, y(0)=-2 ],
          [ x(0)=2, y(0)=1 ], [ x(0)=2, y(0)=0 ], [ x(0)=2, y(0)=-2 ],
          [ x(0)=-2, y(0)=2 ], [ x(0)=-2, y(0)=0 ], [ x(0)=-2, y(0)=-1 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ x(0)=2, y(0)=2], [ x(0)=-2, y(0)=-2 ],
              [ x(0)=2, y(0)=-1 ], [ x(0)=-2, y(0)=1 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Example of Sink` );
```

Example of Sink



Notice how the solutions converge towards the origin along paths that are tangent to the smaller eigenvalue. The different sizes of the eigenvalues can also be seen by noting the relative length of the two straight-line solutions.

>

Harmonic Oscillator Example

```
> A := matrix( [ [ 0, 1 ], [ -1/2, -3/2 ] ] );
```

$$A := \begin{bmatrix} 0 & 1 \\ \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 + \frac{3}{2}\lambda + \frac{1}{2}$$

```
> eigenvals( A );
```

$$-1, \frac{-1}{2}$$

With two negative eigenvalues, the origin is a sink for this system.

```
>
> eigenvects( A );
```

$$[-1, 1, \{[1, -1]\}], \left[\frac{-1}{2}, 1, \{[-2, 1]\}\right]$$

```
>
```

The straight-line solutions are:

```
> Y1 := exp(-t)   * matrix( 2, 1, [ -1, 1 ] );
```

$$Y1 := \mathbf{e}^{(-t)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

```
> Y2 := exp(-t/2) * matrix( 2, 1, [ -2, 1 ] );
```

$$Y2 := \mathbf{e}^{(-1/2\,t)} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

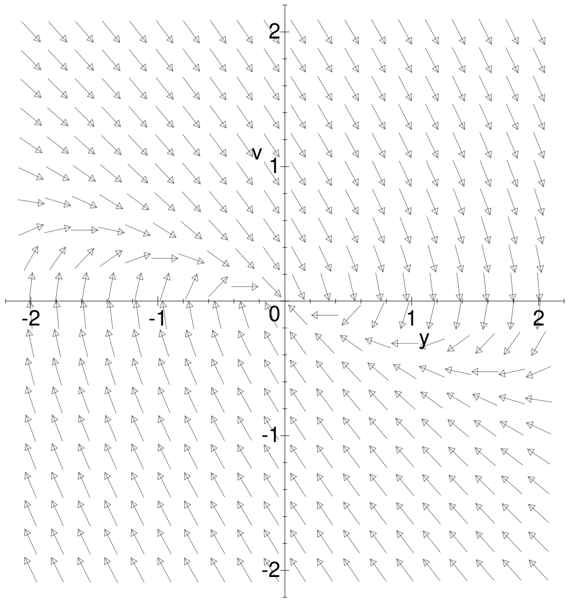
```
>
> ode1 := diff( y(t), t ) = v(t);
> ode2 := diff( v(t), t ) = -1/2*y(t) - 3/2*v(t);
```

```
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = \mathrm{v}(t)$$

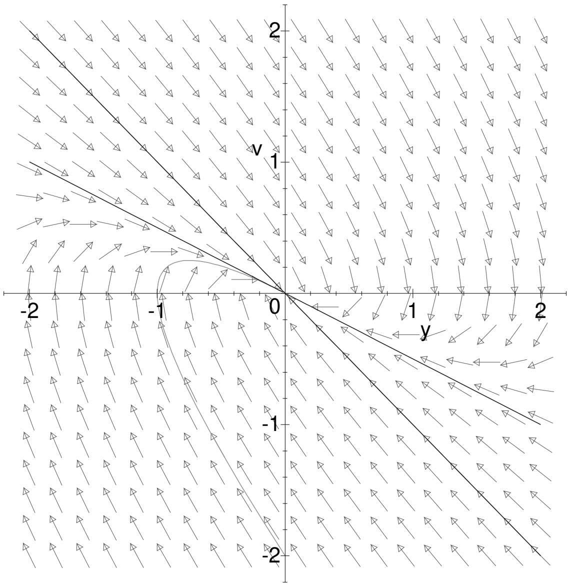
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{v}(t) = -\frac{1}{2}\,\mathrm{y}(t) - \frac{3}{2}\,\mathrm{v}(t)$$

```
> VARS := { y(t), v(t) }:
> DOMAIN := t=0..10:
> RANGE := y=-2..2, v=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ y(0)=0, v(0)=-2 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ y(0)=-2, v(0)=2 ], [ y(0)=2, v(0)=-2 ],
              [ y(0)=-2, v(0)=1 ], [ y(0)=2, v(0)=-1 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Harmonic Oscillator (#16,
  p. 276)` );
```

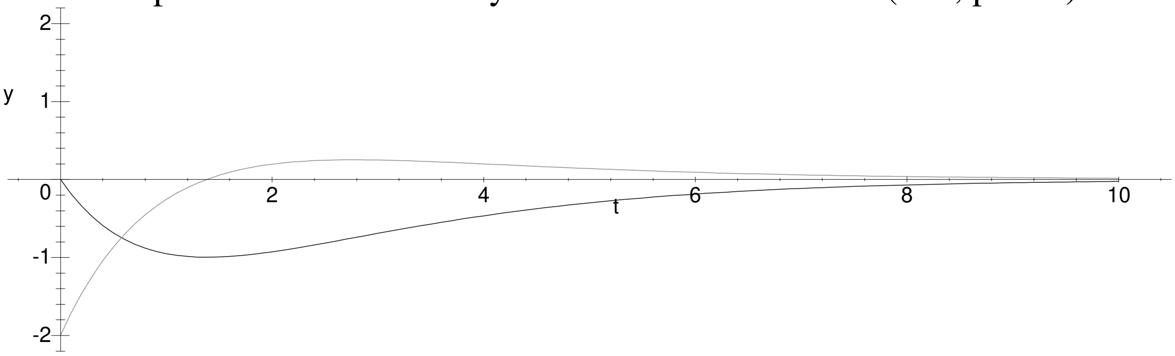
Harmonic Oscillator (#16, p. 276)



Notice how the solution initially moves (essentially) parallel to the eigenvector for the smaller eigenvalue and then bends until it becomes tangent to the eigenvector for the larger eigenvalue as the solution approaches the equilibrium solution. (Recall that the uniqueness theorem guarantees that the solution cannot cross the straight-line solution.)

```
>
> yPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, scene=[t,y], arrows=NONE,
  stepsize=0.1, linecolor=BLUE ):
> vPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, scene=[t,v], arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> display( [ yPLOT, vPLOT ], title=`Displacement and Velocity for Harmonic
  Oscillator (#16, p. 276)` );
>
```

Displacement and Velocity for Harmonic Oscillator (#16, p. 276)




```
>
```

Section 3.4 -- Complex Eigenvalues

Example with $A = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$

```
> A := matrix( [ [ -1, 2 ], [ -1, -1 ] ] );
```

$$A := \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$$

```
>
```

```
> charpoly( A, lambda );
```

$$\lambda^2 + 2\,\lambda + 3$$

```
> eigenvals( A );
```

$$-1 + I\sqrt{2}, -1 - I\sqrt{2}$$

Since the eigenvalues have a negative real part, the origin is a spiral sink for this system.

```
>
> eigenvects( A );
```

$$\left[-1 + I\sqrt{2}, 1, \left\{\left[1, \frac{1}{2} I\sqrt{2}\right]\right\}\right], \left[-1 - I\sqrt{2}, 1, \left\{\left[1, -\frac{1}{2} I\sqrt{2}\right]\right\}\right]$$

```
>
```

The linearly independent solutions are found by the usual steps: exp(eigenvalue * t) * eigenvector, but recall that the two solutions are the real and imaginary parts of this expression. These calculations are messy - by hand. With Maple it's a little better, but there are some extra commands that must be used.

We know that the two solutions will consist of the product of an exponential formed with the real part of the eigenvalues and a vector with trigonometric terms formed using the imaginary part of the eigenvalues. Let's begin by looking at the part of the solution that leads to complex-valued expressions:

```
> assume( t, real );
> mess := exp( I*sqrt(2)*t ) * matrix( 2, 1, [ 2, I*sqrt(2) ] );
```

$$mess := \mathbf{e}^{(I\sqrt{2}\,t\sim)} \begin{bmatrix} 2 \\ I\sqrt{2} \end{bmatrix}$$

```
> mess2 := convert( evalc( evalm( mess ) ), trig );
```

$$mess2 := \begin{bmatrix} 2\cos(\sqrt{2}\,t\sim) + 2\,I\sin(\sqrt{2}\,t\sim) \\ I\,(\cos(\sqrt{2}\,t\sim) + I\sin(\sqrt{2}\,t\sim))\sqrt{2} \end{bmatrix}$$

```
> messRE := map( Re, mess2 );
```

$$messRE := \begin{bmatrix} 2\cos(\sqrt{2}\,t\sim) \\ -\sqrt{2}\,\sin(\sqrt{2}\,t\sim) \end{bmatrix}$$

```
> messIM := map( Im, mess2 );
```

$$messIM := \begin{bmatrix} 2\sin(\sqrt{2}\,t\sim) \\ \sqrt{2}\,\cos(\sqrt{2}\,t\sim) \end{bmatrix}$$

```
>
```

Using the real and imaginary parts we can construct the two linearly independent solutions for this system:

```
> Y1 := exp(-t) * evalm( messRE );
```

$$Y1 := \mathbf{e}^{(-t\sim)} \begin{bmatrix} 2\cos(\sqrt{2}\,t\sim) \\ -\sqrt{2}\,\sin(\sqrt{2}\,t\sim) \end{bmatrix}$$

```
> Y2 := exp(-t) * evalm( messIM );
```

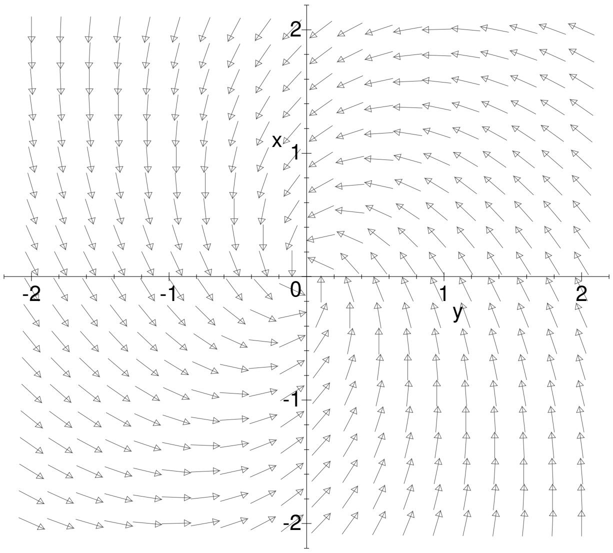
$$Y2 := \mathbf{e}^{(-t\sim)} \begin{bmatrix} 2\sin(\sqrt{2}\,t\sim) \\ \sqrt{2}\,\cos(\sqrt{2}\,t\sim) \end{bmatrix}$$

```
> t := 't':
>
> ode1 := diff( x(t), t ) = -x(t) + 2*y(t);
> ode2 := diff( y(t), t ) = -x(t) - y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = -\mathrm{x}(t) + 2\,\mathrm{y}(t)$$

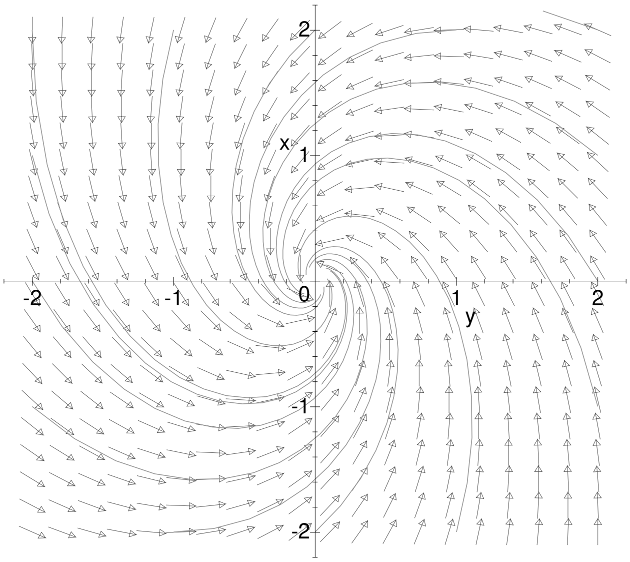
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = -\mathrm{x}(t) - \mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..3:
> RANGE := x=-2..2, y=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=2, y(0)=2 ], [ x(0)=1, y(0)=2 ], [ x(0)=0, y(0)=2 ], [
  x(0)=-1, y(0)=2 ],
          [ x(0)=2, y(0)=-2 ], [ x(0)=1, y(0)=-2 ], [ x(0)=0, y(0)=-2 ], [
  x(0)=-1, y(0)=-2 ],
          [ x(0)=2, y(0)=1 ], [ x(0)=2, y(0)=0 ], [ x(0)=2, y(0)=-1 ],
          [ x(0)=-2, y(0)=1 ], [ x(0)=-2, y(0)=0 ], [ x(0)=-2, y(0)=-1 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> display( [ dirPLOT, solPLOT ], title=`Example of Spiral Sink` );
```

Example of Spiral Sink




```
>
```

```
>
```

Examples of spiral sources and centers can be processed in a similar way.

```
>
```

Section 3.5 -- Special Cases: Repeated and Zero Eigenvalues

Example with $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

```
> A := matrix( [ [ 2, 1 ], [ -1, 4 ] ] );
```

$$A := \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 - 6\lambda + 9$$

```
> eigenvals( A );
```

$$3, 3$$

Notice that the eigenvalues are positive - so the origin is a source - but the eigenvalue is repeated.

```
>
> eigenvects( A );
```

$$[3, 2, \{[1, 1]\}]$$

Note that there is only one eigenvector for this matrix.

```
>
```

The straight-line solutions are:

```
> Y1 := exp(3*t) * matrix( 2, 1, [ 1, 1 ] );
```

$$Y1 := \mathbf{e}^{(3\,t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>
> V1 := matrix( 2, 1, [ 1, 1 ] );
```

$$V1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
> V2 := matrix( 2, 1, [ v1, v2 ] );
```

$$V2 := \begin{bmatrix} v1 \\ v2 \end{bmatrix}$$

```
> eqn := evalm( A &* V2 - ( 3*V2 + V1 ) );
```

$$eqn := \begin{bmatrix} -v1 + v2 - 1 \\ -v1 + v2 - 1 \end{bmatrix}$$

```
> soln := solve( { -v1+v2-1 = 0 }, { v1, v2 } );
```

$$soln := \{ v2 = v2, v1 = v2 - 1 \}$$

```
> V2 := subs( soln , v2=1, evalm(V2) );
```

$$V2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
>
> Y2 := exp(3*t) * evalm( t * V1 + V2 );
```

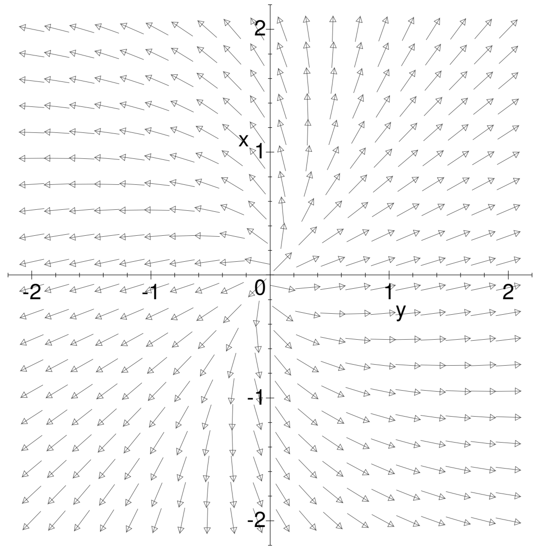
$$Y2 := \mathbf{e}^{(3\,t)} \begin{bmatrix} t \\ t + 1 \end{bmatrix}$$

```
>
> ode1 := diff( x(t), t ) = 2*x(t) + y(t);
> ode2 := diff( y(t), t ) =  -x(t) + 4*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = 2\,\mathrm{x}(t) + \mathrm{y}(t)$$

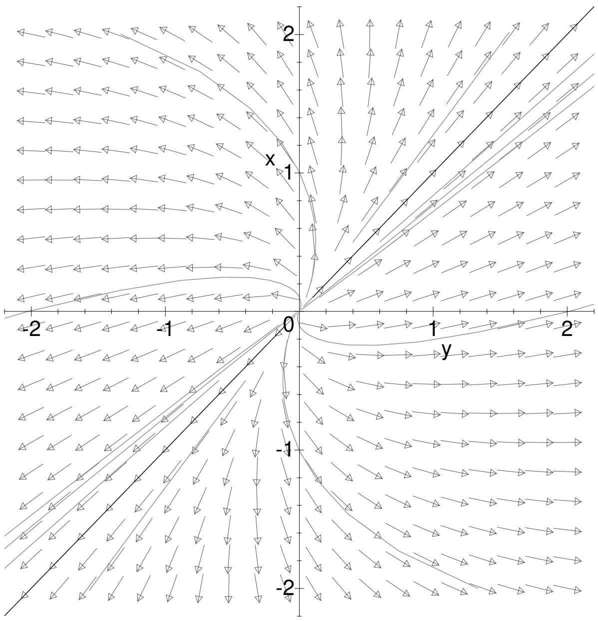
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = -\mathrm{x}(t) + 4\,\mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..3:
> RANGE := x=-2..2, y=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=0.5, y(0)=0.4 ], [ x(0)=0.1, y(0)=0.05 ], [ x(0)=0.01,
  y(0)=0.005 ],
          [ x(0)=0.5, y(0)=0.6 ], [ x(0)=0.1, y(0)=0.15 ], [ x(0)=0.01,
  y(0)=0.015 ],
          [ x(0)=-0.5, y(0)=-0.4 ], [ x(0)=-0.1, y(0)=-0.05 ], [ x(0)=-0.01,
  y(0)=-0.005 ],
          [ x(0)=-0.5, y(0)=-0.6 ], [ x(0)=-0.1, y(0)=-0.15 ], [ x(0)=-0.01,
  y(0)=-0.015 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ x(0)=0.1, y(0)=0.1 ], [ x(0)=-0.1, y(0)=-0.1 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Example of Repeated
  Eigenvalues (Source)` );
```

Example of Repeated Eigenvalues (Source)




```
>
```

Example with $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

```
> A := matrix( [ [ 2, 4 ], [ 3, 6 ] ] );
```

$$A := \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

```
>
> charpoly( A, lambda );
```

$$\lambda^2 - 8\,\lambda$$

```
> eigenvals( A );
```

$$0, 8$$

Since zero is an eigenvalue, there will be a line of equilibrium solutions for this system. Since the other eigenvalue is positive, all other solutions will converge (along straight lines) to an equilibrium solution - not necessarily the origin.

```
>
> eigenvects( A );
```

$$[0, 1, \{[-2, 1]\}], \left[8, 1, \left\{\left[1, \frac{3}{2}\right]\right\}\right]$$

```
>
```

The straight-line solutions are:

```
> Y1 := exp(8*t) * matrix( 2, 1, [ 2, 3 ] );
```

$$Y1 := \mathbf{e}^{(8\,t)} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
> Y2 := exp(0*t) * matrix( 2, 1, [ -2, 1 ] );
```

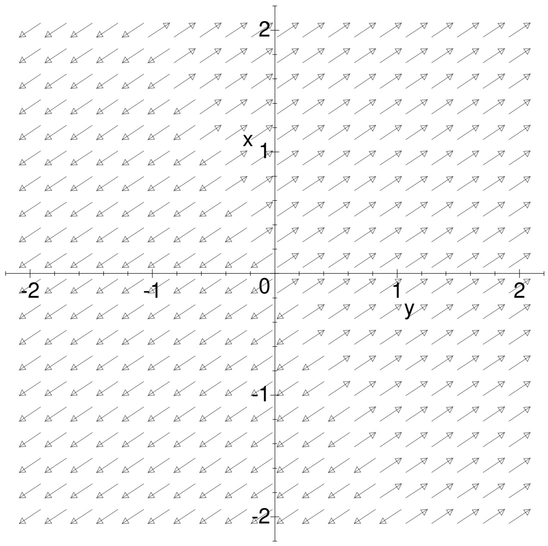
$$Y2 := \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

```
>
> ode1 := diff( x(t), t ) = 2*x(t) + 4*y(t);
> ode2 := diff( y(t), t ) = 3*x(t) + 6*y(t);
> MODEL := { ode1, ode2 }:
```

$$ode1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = 2\,\mathrm{x}(t) + 4\,\mathrm{y}(t)$$

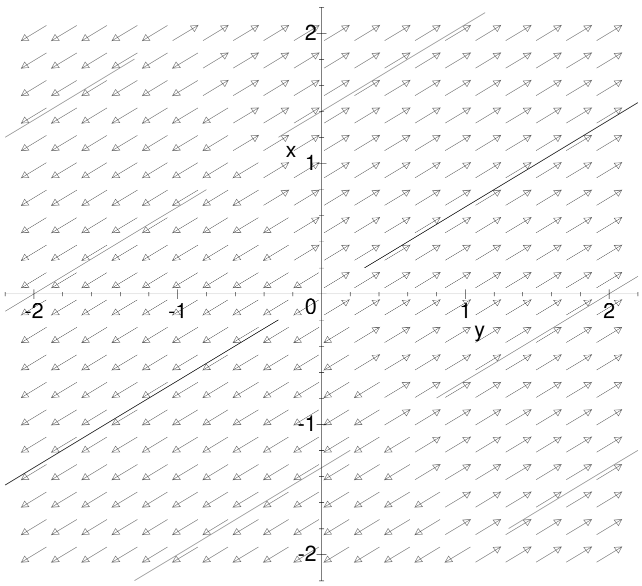
$$ode2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = 3\,\mathrm{x}(t) + 6\,\mathrm{y}(t)$$

```
> VARS := { x(t), y(t) }:
> DOMAIN := t=0..1:
> RANGE := x=-2..2, y=-2..2:
> dirPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, arrows=MEDIUM ):
> dirPLOT;
```




```
>
> IC := [ [ x(0)=-0.8, y(0)=0.8 ], [ x(0)=-1.2, y(0)=0.2 ],
          [ x(0)=1.2, y(0)=-0.3 ], [ x(0)=0.8, y(0)=-0.8 ],
          [ x(0)=-1.8, y(0)=1.3 ], [ x(0)=-2.2, y(0)=0.7 ],
          [ x(0)=2.2, y(0)=-0.7 ], [ x(0)=1.8, y(0)=-1.3 ] ]:
> solPLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, IC, arrows=NONE,
  stepsize=0.1, linecolor=GREEN ):
> #solPLOT;
>
> ICline := [ [ x(0)=0.2, y(0)=0.3], [ x(0)=-0.2, y(0)=-0.3 ],
              [ x(0)=-2, y(0)=1 ], [ x(0)=2, y(0)=-1 ],
              [ x(0)=-1, y(0)=0.5 ], [ x(0)=1, y(0)=-0.5 ] ]:
> linePLOT := DEplot( MODEL, VARS, DOMAIN, RANGE, ICline, arrows=NONE,
  linecolor=BLUE ):
> #linePLOT;
>
> display( [ dirPLOT, solPLOT, linePLOT ], title=`Example of Sink` );
```

Example of Sink



Notice how the solutions travel along paths that are parallel to the eigenvector for the non-zero eigenvalue and the line of equilibrium solutions conicides with the direction of the zero eigenvalue.

>

>