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BDH1-1.mws

Modelling Via Differential Equations

Section 1.1 - Blanchard, Devaney, Hall

18 January 1998

Doug Meade (modified by M^2)

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Introduction

The purpose of this worksheet is to introduce you to some of the ways Maple can be used to assist with the analysis of differential equations. You are not expected to currently understand everything that is presented in this worksheet. As the course progresses you will be expected to become proficient at these types of "computations".

This worksheet is intended to be use while reading section 1.1 of the text. The definitions are not repeated in this worksheet. There are a few places where this worksheet differs from the text. These are clearly identified. There are a number of questions for you to answer as you work through this worksheet. Please take the time to do what is asked. Ask questions as they arise.

Getting Started

Every Maple worksheet should begin by re-initializing the Maple "kernel" and loading the additional packages that we are most likely to use.

```
> restart;
> with( plots ):
> with( DEtools ):
>
```

Unlimited Population Growth

One model for unlimited population growth is represented by the following first-order ordinary differential equation.

```
> MODEL := diff( P(t), t ) = k * P(t);
```

$$MODEL := \frac{\partial}{\partial t} P(t) = k\, P(t)$$

Question

Equilibrium Solution

(See the text for a definition of equilibrium solution.)

```
> EQUILeqn := rhs( MODEL ) = 0;
```

$$EQUILeqn := k\, P(t) = 0$$

```
> EQUILsol := solve( EQUILeqn, { P(t) } );
```

$$EQUILsol := \{P(t) = 0\}$$

Qualitative Analysis

```
> InitCond := P(0) = p[0];
```

$$InitCond := P(0) = p_0$$

The text provides a method by which we can determine that solutions to this model are increasing if $k > 0$ and $p_0 > 0$ and decreasing if $k < 0$ and $p_0 > 0$. This type of analysis can be done for other equations but is often quite difficult.

```
>
```

Analytic Solution

We will learn specific techniques for solving special classes of differential equations. The Maple command for finding explicit solutions to differential equations is dsolve. *The arguments are the initial value problem (a set containing the differential equation and the initial condition) and the function for which we are trying to solve.*

```
> SOLN := dsolve( { MODEL, InitCond }, P(t) );
```

$$SOLN := P(t) = e^{(k\, t)}\, p_0$$

Warning

```
>
```

The U.S. Population

The computations used to determine values for the ***parameters*** *k and p_0 from the census data are easily reproduced in Maple. The idea is that two data points can be used to construct a system of two equations for the two unknown constants. While it is possible to exactly duplicate the steps shown in the text to determine k and p_0, it is simpler to ask Maple to solve the system.*

```
>
> EQ1790 := subs( P(t)= 3.9, t= 0, SOLN );
```

$$EQ1790 := 3.9 = e^0 p_0$$

```
> EQ1800 := subs( P(t)= 5.3, t=10, SOLN );
```

$$EQ1800 := 5.3 = e^{(10\,k)} p_0$$

```
> PARAMETERS := solve( { EQ1790, EQ1800 }, { k, p[0] } );
```

$$PARAMETERS := \{p_0 = 3.900000000, k = .03067302674\}$$

Thus, this model predicts the U.S. population is given by the following function.

```
> solP := subs( PARAMETERS, rhs(SOLN) );
```

$$solP := 3.900000000\, e^{(.03067302674\, t)}$$

```
>
```

Here is the actual data from the text.

```
> data:= [1790,3.9], [1800,5.3], [1810,7.2], [1820,9.6], [1830,12], [1840,17], [1850,23], [1860,31],
  [1870,38], [1880, 50], [1890,62], [1900,75], [1910,91], [1920,105], [1930,122], [1940,131],
  [1950,151], [1960,179], [1970,203], [1980,226], [1990,249], [2000,`NA`], [2010,'NA'], [2020,'NA'],
  [2030,'NA'], [2040,'NA'], [2050,'NA'];
```

data := [1790, 3.9], [1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12], [1840, 17], [1850, 23], [1860, 31], [1870, 38], [1880, 50], [1890, 62], [1900, 75], [1910, 91], [1920, 105], [1930, 122], [1940, 131], [1950, 151], [1960, 179], [1970, 203], [1980, 226], [1990, 249], [2000, NA], [2010, NA], [2020, NA], [2030, NA], [2040, NA], [2050, NA]

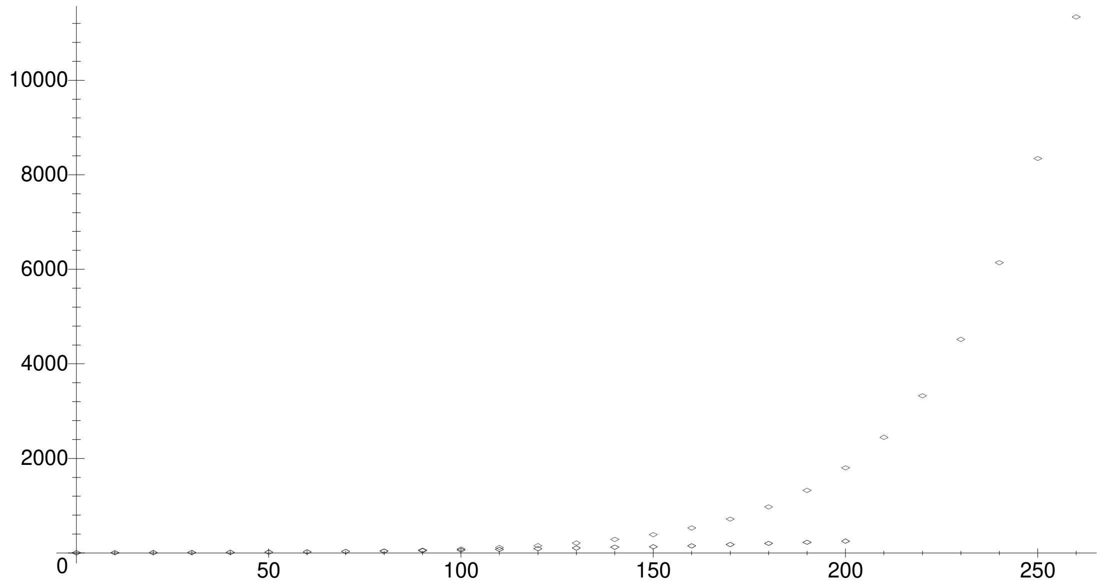
```
> shifted_data:= seq( [ 10*(k-1), data[k,2] ], k = 1 .. nops([data]) );
```

shifted_data := [0, 3.9], [10, 5.3], [20, 7.2], [30, 9.6], [40, 12], [50, 17], [60, 23], [70, 31], [80, 38], [90, 50], [100, 62], [110, 75], [120, 91], [130, 105], [140, 122], [150, 131], [160, 151], [170, 179], [180, 203], [190, 226], [200, 249], [210, NA], [220, NA], [230, NA], [240, NA], [250, NA], [260, NA]

```
>
```

To evaluate the reasonableness of this prediction we will often plot the function. However, since the text provides a table with census values from 1790 through 2050, let's produce predicted populations for each decade (and a few decades into the next millenium). Notice that t in the formula is not "date" -- the year 1790 corresponds to the value t = 0. We produce a second list, called shifted_data, that uses t = 0 as the starting time.

```
> soln_points:= NULL:
> for t from 0 to 260 by 10 do
>   printf( `%4d  %10.4f  %10.4a\n`, t+1790, solP, shifted_data[t/10+1,2] );
>   soln_points:= soln_points, [t, solP]:
> od:
1790      3.9000         3.9
1800      5.3000         5.3
1810      7.2025         7.2
1820      9.7880         9.6
1830     13.3017          12
1840     18.0767          17
1850     24.5658          23
1860     33.3843          31
1870     45.3685          38
1880     61.6546          50
1890     83.7871          62
1900    113.8645          75
1910    154.7389          91
1920    210.2863         105
1930    285.7737         122
1940    388.3591         131
1950    527.7701         151
1960    717.2260         179
1970    974.6918         203
1980   1324.5812         226
1990   1800.0719         249
2000   2446.2515          NA
2010   3324.3931          NA
2020   4517.7650          NA
2030   6139.5269          NA
2040   8343.4596          NA
2050  11338.5477          NA
> t:= 't':
> plot( [ [soln_points], [shifted_data] ], color=[red, blue], style = POINT );
```




```
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```

Question

Question

```
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```

Logistic Population Growth

The logistic population model is motivated and derived in the text (pp. 8-9). The resulting differential equation is

```
> MODEL := diff( P(t), t ) = k * ( 1 - P(t)/N ) * P(t);
```

$$MODEL := \frac{\partial}{\partial t} P(t) = k \left(1 - \frac{P(t)}{N}\right) P(t)$$

with the initial condition

```
> InitCond := P(0) = p[0];
```

$$InitCond := P(0) = p_0$$

```
>
```

Equilibrium Solution

The equilibrium solutions are found in the same manner as in the first example: determine all constant populations (i.e., for which the derivative, dP/dt, and hence the RHS of the model, is zero).

```
> EQUILeqn := rhs( MODEL ) = 0;
```

$$EQUILeqn := k \left(1 - \frac{P(t)}{N}\right) P(t) = 0$$

```
> EQUILsol := solve( EQUILeqn, { P(t) } );
```

$$EQUILsol := \{P(t) = 0\}, \{P(t) = N\}$$

Note that there are now two equilibria. In addition to the zero solution, the carrying capacity of the population, N, is an equilibrium solution.

```
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```

Qualitative Analysis

The text gives a good description of the qualitative analysis of the logistic model. Assume $k > 0$. The basic idea is that (for equations of this type - we'll be more specific shortly) the solutions are either always increasing or always decreasing in regions between the equilibria. In this case, for $0 < P < N$, the RHS of the model is positive $k\left(1 - \frac{P}{N}\right)P > 0$, and so the solutions are increasing (to the carrying capacity). By the same reasoning, for $P > N$ or $P < 0$, $k\left(1 - \frac{P}{N}\right)P < 0$ and solutions are decreasing (to the carrying capacity).

Questions

```
>
```

Analytic Solution

We will learn specific techniques for solving special classes of differential equations. The Maple command for finding explicit solutions to differential equations is dsolve. The arguments are the initial value problem (a set containing the differential equation and the initial condition) and the function for which we are trying to solve.

```
> SOLNlog := dsolve( { MODEL, InitCond }, P(t) );
```

$$SOLNlog := P(t) = \frac{N}{1 - \frac{e^{(-k\,t)}\,(-N+p_0)}{p_0}}$$

```
>
```

The U.S. Population

To conclude this investigation of the models, lets determine the values of the parameters that fit the U.S. population in 1790, 1900, and 1990. (Why are three years used for this fit?)

```
> EQ1790 := subs( P(t)=3.9, t=0, SOLNlog );
```

$$EQ1790 := 3.9 = \frac{N}{1 - \frac{e^{0}\,(-N+p_0)}{p_0}}$$

```
> EQ1900 := subs( P(t)=75, t=110, SOLNlog );
```

$$EQ1900 := 75 = \frac{N}{1 - \frac{e^{(-110\,k)}\,(-N+p_0)}{p_0}}$$

```
> EQ1990 := subs( P(t)=249, t=200, SOLNlog );
```

$$EQ1990 := 249 = \frac{N}{1 - \frac{e^{(-200\,k)}\,(-N+p_0)}{p_0}}$$

```
>
```

The solution(s) to this system of three equations and three unknowns can be found using the solve command.

```
> solve( { EQ1790, EQ1900, EQ1990 }, { k, N, p[0] } );
```

$\{p_0 = 3.900000000, k = -.0001150272941 + .2789755232\,I, N = 1.161038771 + 2.624992784\,I\}$,

$\{p_0 = 3.900000000, k = -.0001150272941 - .2789755232\,I, N = 1.161038771 - 2.624992784\,I\}$,

$\{p_0 = 3.900000000, k = .02947747973 + .2859704219\,I, N = 284.7715405 - 34.53016274\,I\}$,

$\{p_0 = 3.900000000, k = .02947747973 - .2859704219\,I, N = 284.7715405 + 34.53016274\,I\}$,

$\{p_0 = 3.900000000, k = -.0006236785324 + .2091206755\,I, N = 5.795376720 + 2.777691875\,I\}$,

$\{p_0 = 3.900000000, k = -.0006236785324 - .2091206755\,I, N = 5.795376720 - 2.777691875\,I\}$,

$\{p_0 = 3.900000000, N = 235.0078808 - 33.24516748\,I, k = .03012074910 + .2290429768\,I\}$,

$\{p_0 = 3.900000000, k = .03012074910 - .2290429768\,I, N = 235.0078808 + 33.24516748\,I\}$,

$\{p_0 = 3.900000000, N = 219.0625755 - 6.857716608\,I, k = .03051848785 + .1715023717\,I\}$,

$\{p_0 = 3.900000000, k = .03051848785 - .1715023717\,I, N = 219.0625755 + 6.857716608\,I\}$,

$\{p_0 = 3.900000000, k = -.0007688614606 + .1397466913\,I, N = 6.984565519 - 1.024798866\,I\}$,

$\{p_0 = 3.900000000, N = 6.984565519 + 1.024798866\,I, k = -.0007688614606 - .1397466913\,I\}$,

$\{p_0 = 3.900000000, N = 223.8885768 + 20.48162777\,I, k = .03038028130 + .1138409154\,I\}$,

$\{p_0 = 3.900000000, N = 223.8885768 - 20.48162777\,I, k = .03038028130 - .1138409154\,I\}$,

$\{p_0 = 3.900000000, k = .02978640096 + .05654903646\,I, N = 255.2974247 + 41.61353030\,I\}$,

$\{p_0 = 3.900000000, N = 255.2974247 - 41.61353030\,I, k = .02978640096 - .05654903646\,I\}$,

$\{p_0 = 3.900000000, k = .02934545722, N = 302.9459184\}$,

$\{p_0 = 3.900000000, N = 3.609018089 - 3.555570634\,I, k = -.0003731412174 + .07020511101\,I\}$,

$\{p_0 = 3.900000000, k = -.0003731412174 - .07020511101\,I, N = 3.609018089 + 3.555570634\,I\}$

Note that while there are many solutions, all but one of them involves complex numbers. Since we know our answer should be real-valued (WHY?) we can assume that these are the parameter values that we should use for this problem. There is a fancy way to have Maple select all real-valued solutions but it's simpler to simply copy and paste the portion of the output that interests us.

```
> PARAMETERS := {p[0] = 3.900000000, k = .2934545722e-1, N = 302.9459184};
```

$$PARAMETERS := \{p_0 = 3.900000000, k = .02934545722, N = 302.9459184\}$$

(optional) Fancy method of extracting real-valued solution from multiple solutions of a system of equations

```
>
```

The logistic model for the U.S. population based on the 1790, 1900, and 1990 populations is

```
> solPlog := subs( PARAMETERS, rhs( SOLNlog ) );
```

$$solPlog := \frac{302.9459184}{1 + 76.67844062\,e^{(-.02934545722\,t)}}$$

```
> soln_points:= NULL:
```

A table of predicted populations for 1790 - 2050 is easily created as before

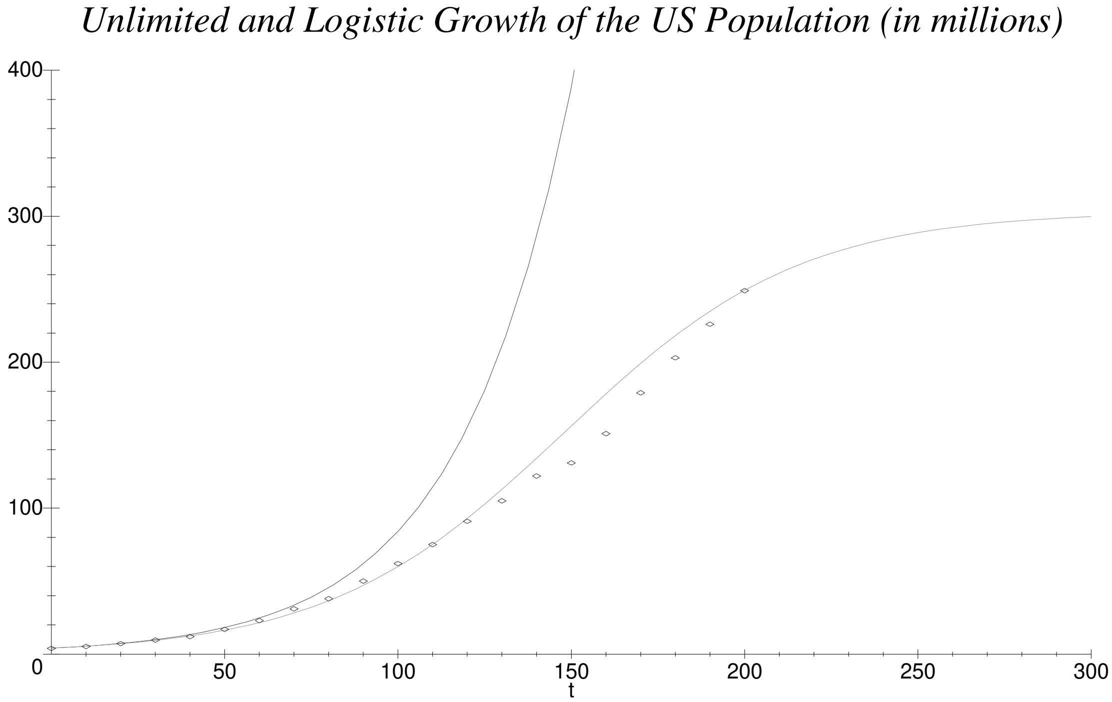
```
> for t from 0 to 260 by 10 do
>   printf( `%4d  %10.4f  %10.4f %10.4a\n`, t+1790, solP, solPlog, shifted_data[t/10+1,2] );
>   soln_points:= soln_points, [t, solPlog]:
> od:
1790      3.9000      3.9000        3.9
1800      5.3000      5.2072        5.3
```

1810	7.2025	6.9424	7.2
1820	9.7880	9.2380	9.6
1830	13.3017	12.2611	12
1840	18.0767	16.2189	17
1850	24.5658	21.3604	23
1860	33.3843	27.9728	31
1870	45.3685	36.3677	38
1880	61.6546	46.8528	50
1890	83.7871	59.6839	62
1900	113.8645	74.9999	75
1910	154.7389	92.7478	91
1920	210.2863	112.6205	105
1930	285.7737	134.0360	122
1940	388.3591	156.1820	131
1950	527.7701	178.1284	151
1960	717.2260	198.9776	179
1970	974.6918	218.0049	203
1980	1324.5812	234.7436	226
1990	1800.0719	248.9999	249
2000	2446.2515	260.8112	NA
2010	3324.3931	270.3747	NA
2020	4517.7650	277.9753	NA
2030	6139.5269	283.9271	NA
2040	8343.4596	288.5338	NA
2050	11338.5477	292.0674	NA

```
> t := 't':
>
```

Note that the logistic model does a much better job of following the actual U.S. populations (as reported in Table 1.1 on p. 7). A final comparison of the two solutions can be performed by looking at the plot of the solutions to each model. Here we want to overlay two different kinds of plots (not just two curves in one kind of plot). The way to do this is to give the plots names, end the commands with colons, and then "display" the two together

```
> plotA:= plot( [ solP, solPlog ], t=0..300, 0..400, color=[red, green], title=`Unlimited and Logistic
  Growth of the US Population (in millions)` ):
> plotB:= plot( [ [shifted_data] ], color = blue, style = POINT):
> display([plotA, plotB]);
```



Note how one solution appears to level off around 300 (million) and the other appears to continue to grow forever. Which curve corresponds with which model?

```
>
> restart:
>
```