MATH 241 | University of South Carolina
Prof. Meade | Fall 2000

Final Exam
December 11, 2000

Name: ______________________
SS #: ______________________

Instructions:

1. There are a total of 10 problems on 9 pages. Check that your copy of the exam has all of the problems.
2. You must show all of your work to receive credit for a correct answer.
3. I will provide paper for all of your work. Please begin each problem on a new piece of paper and organize your papers by problem number. It is your responsibility to clearly label your work.

Problem	Points	Score
1	10	
2	18	
3	16	
4	12	
5	10	
6	15	
7	21	
8	18	
9	12	
10	18	
Total	150	

Happy Holidays!

1. (10 points) Find the length of the curve

$$\begin{aligned} x &= \cos t + t \sin t \\ y &= \sin t - t \cos t \end{aligned}$$

from 0 to 2π.

2. (18 points) Consider the curve $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j}$.

   (a) Find the unit tangent vector $\mathbf{T}(t)$ to the curve.

   (b) Find the point on the curve when $t = 1$.

   (c) Find the tangent line to the curve when $t = 1$.

3. (16 points) Write an equation of the plane through the point $(12, 25, 0)$ that satisfies each condition.

   (a) Parallel to the xy-plane.

   (b) Perpendicular to the x-axis.

   (c) Parallel to the x- and z-axes.

   (d) Parallel to the plane $4x - 2y + z = 5$.

4. (12 points) For $f(x, y) = \frac{1}{2}x^2 + y^2$,

   (a) find the equation of its level curve that goes through the point $(4, 1)$;

   (b) find a normal vector to the level curve in (a) at $(4, 1)$.

5. (10 points) Find and classify the extrema of $f(x, y) = x^2y - 6y^2 - 3x^2$.

6. (15 points) A triangle has vertices A, B, and C. Let the side between vertices A and B be $c = AB$, the side $b = AC$, and the included angle α.

   (a) Use the cross-product to show that the area of the triangle is $A = \frac{1}{2}bc\sin\alpha$.

   (b) Find the rate at which the area is changing when the side $c = 10$ inches and increasing at the rate of 3 inches per second, the side $b = 8$ inches and decreasing at 1 inch per second, and the $\alpha = \frac{\pi}{6}$ radians and decreasing at 0.1 radian per second.

7. (21 points) Evaluate each of the following multiple integrals.

(a) $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2xy^2 \; dx \; dy$

(b) $\int \int_S \frac{1}{x^2+y^2} \; dA$ where S is the region between the circles $x^2+y^2=4$ and $x^2+y^2=9$.

(c) $\int_1^4 \int_3^x \int_0^{\sqrt{3}y} \frac{z}{y^2+z^2} \; dz \; dy \; dx$

8. (18 points) Let $\mathbf{F}(x, y, z) = 2xyz\mathbf{i} - 3y^2\mathbf{j} + 2y^2z\mathbf{k}$. Find

   (a) $\text{div}\mathbf{F}$

   (b) $\text{curl}\mathbf{F}$

   (c) $\nabla(\text{div}\mathbf{F})$

9. (12 points)

   (a) Find a function f satisfying $\nabla f = (yz - e^{-x})\mathbf{i} + (xz + e^{y})\mathbf{j} + xy\mathbf{k}$

   (b) Use (a) to evaluate $\displaystyle\int_{(0,0,0)}^{(1,1,4)} (yz - e^{-x})\ dx + (xy + e^{y})\ dy + xy\ dz.$

10. (18 points) Use Green's Theorem to evaluate $\int_C xy\ dx + (x^2 + y^2)\ dy$ if

(a) C is the square path $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$;

(b) C is the triangular path $(0,0)$ to $(2,0)$ to $(2,1)$ to $(0,0)$;

(c) C is the circle $x^2 + y^2 = 1$ traversed in the *clockwise* direction.

(d) Explain how the results in (a), (b), and (c) tell you the vector field $\mathbf{F} = xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is *not* conservative.