Exam2 - Key



1. (12 points) Find each limit, or explain why it does not exist.

DNE (a) $\lim_{(x,y)\to(0,0)} \dfrac{xy\cos(y)}{3x^2+y^2}$ along $y=mx$: $\lim_{x\to 0} \dfrac{x(mx)\cos(mx)}{3x^2+m^2x^2} = \lim_{x\to 0} \dfrac{m}{3+m^2}\cos(mx)$

$= \dfrac{m}{3+m^2}$ (depends on $m$)

(b) $\lim_{(x,y)\to(1,0)} \ln\left(\dfrac{1+y^2}{x^2+xy}\right) = \ln\left(\dfrac{1+0^2}{1^2+1\cdot 0}\right) = \ln 1 = 0$

2. (14 points) Find the linearization $L(x,y)$ of the function $f(x,y) = y\sqrt{x}$ at $(4,1)$.

$f_x = \frac{1}{2}x^{-1/2}y \quad f_x(4,1) = \frac{1}{2}4^{-1/2}\cdot 1 = \frac{1}{4} \qquad f(4,1) = 1\cdot\sqrt{4} = 2$

$f_y = x^{1/2} \quad f_y(4,1) = 4^{1/2} = 2$

$L(x,y) = 2 + \frac{1}{4}(x-4) + 2(y-1) = \frac{1}{4}x + 2y - 1$

3. (14 points) Find the the equation of the tangent plane to the surface $(x-1)^2 + 2(y-4)^2 + (z-3)^2 = 10$ at the point $(3,3,5)$.

This surface is a level surface: $F(x,y,z) = 10$, $F(x,y,z) = (x-1)^2 + 2(y-4)^2 + (z-3)^2$

$\vec{n} = \nabla F(3,3,5) = \langle 2(x-1), 4(y-4), 2(z-3)\rangle \big|_{(3,3,5)} = \langle 4,-4,4\rangle$

Tangent plane: $4(x-3) - 4(y-3) + 4(z-5) = 0$

or $x - y + z = 5$.

4. (14 points) If $z = f(x,y)$, where $f$ is differentiable, and $x = g(t)$ and $y = h(t)$ with $g(3) = 2$, $h(3) = 7$, $g'(3) = 5$, $h'(3) = -4$, $f_x(2,7) = -8$, and $f_y(2,7) = 6$. Find $\frac{dz}{dt}$ when $t = 3$.

$$\begin{array}{ccc} & z & \\ x & & y \\ | & & | \\ t & & t \end{array}$$

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = f_x(2,7)\,g'(3) + f_y(2,7)\,h'(3) = (-8)5 + 6(-4)$

$= -40 - 24$

$= -64$

When $t=3$: $x = g(3) = 2$

$y = h(3) = 7$

5. (14 points) Find all points $(x,y)$ at which the direction of fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + 2\mathbf{j}$.

direction of fastest change of $f$ is $\nabla f = \langle 2x-2, 2y-4\rangle$.

we want to know when $\nabla f \parallel \langle 1,2\rangle$, i.e. $\nabla f = k\langle 1,2\rangle$:

$2x - 2 = k \Rightarrow x = \frac{k+2}{2}$

$2y - 4 = 2k \Rightarrow y = k+2 = 2\left(\frac{k+2}{2}\right) = 2x \qquad \therefore \nabla f \parallel \langle 1,2\rangle$ along $y = 2x$.

6. (16 points) Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^3 - 6xy - y^3$.

$f_{xx} = 6x$

$f_{yy} = -6y$

$f_{xy} = -6$

| $x$ | $y$ | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | $D$ | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | -6 | -36 | saddle point |
| -2 | 2 | -12 | -12 | -6 | 108 | local max. |

$\nabla f = \langle 3x^2 - 6y, -6x - 3y^2\rangle = \vec{0}$

$3x^2 - 6y = 0 \Rightarrow y = \frac{1}{2}x^2$

$-6x - 3y^2 = 0 \Rightarrow 0 = -6x - 3\left(\frac{1}{2}x^2\right)^2$

$= -6x - \frac{3}{4}x^4 = -\frac{3}{4}x(x^3+8) \Rightarrow x = 0 \ (y = \frac{1}{2}x^2 = 0)$

$x = -2 \ (y = \frac{1}{2}x^2 = 2)$

7. (16 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\underbrace{\frac{1}{x^2} + \frac{1}{y^2}}_{g(x,y)} = 2$.

$\nabla f = \langle -x^{-2}, -y^{-2}\rangle \qquad \nabla f = \lambda\nabla g \Rightarrow -\dfrac{1}{x^2} = -\dfrac{2\lambda}{x^3} \Rightarrow x = 2\lambda$

$\nabla g = \langle -2x^{-3}, -2y^{-3}\rangle \qquad -\dfrac{1}{y^2} = -\dfrac{2\lambda}{y^3} \Rightarrow y = 2\lambda$

$g(x,y) = 2$: $2 = \dfrac{1}{(2\lambda)^2} + \dfrac{1}{(2\lambda)^2} = \dfrac{2}{4\lambda^2}$

$\Rightarrow \lambda^2 = 1/4 \Rightarrow \lambda = \pm\frac{1}{2}$.

$\lambda = \frac{1}{2}$: $x = y = 1$ $f(x,y) = 1 + 1 = 2$ max

$\lambda = -\frac{1}{2}$: $x = y = -1$ $f(-1,-1) = -1 + -1 = -2$ min